FINAL JEE–MAIN EXAMINATION – JANUARY, 2020

(Held On Thursday 09th JANUARY, 2020) TIME : 9 : 30 AM to 12 : 30 PM

MATHEMATICS — TEST PAPER WITH ANSWER & SOLUTION

1. A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness the melts at a rate of 50 cm³/min. When the thickness of ice is 5 cm, then the rate (in cm/min.) at which of the thickness of ice decreases, is :

(1) $\frac{1}{36\pi}$ (2) $\frac{5}{6\pi}$ (3) $\frac{1}{18\pi}$ (4) $\frac{1}{54\pi}$

NTA Ans. (3)

Sol. Let thickness of ice be 'h'.

Vol. of ice = $v=\frac{4\pi}{3}\left((10+h)^3-10^3\right)$

$$\frac{dv}{dt}=\frac{4\pi}{3}\left(3(10+h)^2\right).\frac{dh}{dt}$$

Given $\frac{dv}{dt}=50cm^3/min$ and h = 5cm

$\Rightarrow$ $50=\frac{4\pi}{3}\left(3(10+5)^2\right)\frac{dh}{dt}$

$\Rightarrow$ $\frac{dh}{dt}=\frac{50}{4\pi\times15^2}=\frac{1}{18\pi}$ cm / min

2. If the number of five digit numbers with distinct digits and 2 at the 10th place is 336 k, then k is equal to :

(1) 8 (2) 6 (3) 4 (4) 7

NTA Ans. (1)

Sol. _ _ _ 2 _

No. of five digits numbers =

No. of ways of filling remaining 4 places

= 8 × 8 × 7 × 6

$\therefore$ $k=\frac{8\times8\times7\times6}{336}=8$

3. Let z be complex number such that $\left|\frac{z-i}{z+2i}\right|=1$ and $|z|=\frac{5}{2}$. Then the value of |z + 3i| is :

(1) $\sqrt{10}$ (2) $2\sqrt{3}$ (3) 7 (4) 15

Sol. $\left|\frac{z-i}{z+2i}\right|=1$

$\Rightarrow$ |z – i| = |z + 2i|

$\Rightarrow$ z lies on perpendicular bisector of (0, 1) and (0, –2).

$\Rightarrow$ $Imz=-\frac{1}{2}$

Let $z=x-\frac{i}{2}$

$\because$ $|z|=\frac{5}{2}$ $\Rightarrow$ $x^2=6$

$\therefore$ $|z+3i|=\left|x+\frac{5i}{2}\right|=\sqrt{x^2+\frac{25}{4}}$

$=\sqrt{6+\frac{25}{4}}=\frac{7}{2}$

4. In a box, there are 20 cards, out of which 10 are lebelled as A and the remaining 10 are labelled as B. Cards are drawn at random, one after the other and with replacement, till a second A-card is obtained. The probability that the second A-card appears before the third B-card is :

(1) $\frac{11}{16}$ (2) $\frac{13}{16}$ (3) $\frac{9}{16}$ (4) $\frac{15}{16}$

NTA Ans. (1)

Sol. A : Event when card A is drawn

B : Event when card B is drawn.

$P(A)=P(B)=\frac{1}{2}$

Required probability = P(AA or (AB)A or (BA)A or (ABB)A or (BAB)A or (BBA)A)

$=\frac{1}{2}\times\frac{1}{2}+\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)\times2+\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)\times3$

1 1 3 11

5. The value of $\int_0^{2\pi} \frac{x\sin^8 x}{\sin^8 x+\cos^8 x}dx$ is equal to :

(1) 2π (2) 4π (3) $2\pi^2$ (4) π^2

NTA Ans. (4)

Sol. $I = \int_0^{2\pi} \frac{x\sin^8 x}{\sin^8 x+\cos^8 x}dx$ ....(1)

$$=\left[\int_0^{\pi} \frac{x\sin^8 x}{\sin^8 x\cos^8 x}dx+\int_0^{\pi}\frac{(2\pi-x)\sin^8 x}{\sin^8 x+\cos^8 x}dx\right]$$

$$=2\pi\int_0^{\pi}\frac{\sin^8 x}{\sin^8 x+\cos^8 x}dx$$

$$I=2\pi\left[\int_0^{\pi/2}\frac{\sin^8 x}{\sin^8 x+\cos^8 x}dx+\int_0^{\pi/2}\frac{\cos^8 xdx}{\sin^8 x+\cos^8 x}dx\right]$$

$$=2\pi\int_0^{\pi/2}1dx=2\pi.\frac{\pi}{2}=\pi^2$$

6. If $f'(x) = \tan^{-1}(\sec x + \tan x)$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$, and $f(0) = 0$, then $f(1)$ is equal to :

(1) $\frac{\pi-1}{4}$ (2) $\frac{\pi+2}{4}$

(3) $\frac{\pi+1}{4}$ (4) $\frac{1}{4}$

NTA Ans. (3)

Sol. $f'(x) = \tan^{-1}(\sec x + \tan x)$

$$f'(x) = \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)=\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right)$$

$$\because\quad -\frac{\pi}{2}<x<\frac{\pi}{2}\Rightarrow 0<\frac{\pi}{4}+\frac{x}{2}<\frac{\pi}{2}$$

$$\Rightarrow\quad f'(x)=\frac{\pi}{4}+\frac{x}{2}$$

$$\therefore\quad f(x)=\frac{\pi}{4}.x+\frac{x^2}{4}+c$$

$\because\quad f(0) = 0 \Rightarrow c = 0$

$$\Rightarrow\quad f(x)=\frac{\pi}{4}x+\frac{x^2}{4}$$

$\pi+1$

7. If the matrices $A=\begin{bmatrix}1&1&2\\1&3&4\\1&-1&3\end{bmatrix}$, B = adjA and C = 3A, then $\frac{|\text{adj}B|}{|C|}$ is equal to :

(1) 72 (2) 2 (3) 8 (4) 16

NTA Ans. (3)

Sol. $A=\begin{bmatrix}1&1&2\\1&3&4\\1&-1&3\end{bmatrix}$

$\Rightarrow\quad |A| = 6$

$$\frac{|\text{adj}B|}{|c|}=\frac{|\text{adj}(\text{adj}A)|}{|9A|}=\frac{|A|^4}{3^3|A|}=\frac{|A|^3}{3^3}$$

$$=\frac{(6)^3}{(3)^3}=8$$

8. The number of real roots of the equation, $e^{4x} + e^{3x} - 4e^{2x} + e^x + 1 = 0$ is :

(1) 4 (2) 2 (3) 3 (4) 1

NTA Ans. (4)

Sol. $e^{4x} + e^{3x} - 4e^x + e^x + 1 = 0$

Divide by e^{2x}

$$\Rightarrow\quad e^{2x}+e^x-4+\frac{1}{e^x}+\frac{1}{e^{2x}}=0$$

$$\Rightarrow\quad \left(e^{2x}+\frac{1}{e^{2x}}\right)+\left(e^x+\frac{1}{e^x}\right)-4=0$$

$$\Rightarrow\quad \left(e^x+\frac{1}{e^x}\right)^2-2+\left(e^x+\frac{1}{e^x}\right)-4=0$$

Let $e^x+\frac{1}{e^x}=t \Rightarrow (e^x-1)^2 = 0 \Rightarrow x = 0.$

$\therefore$ Number of real roots = 1

9. Negation of the statement :

$\sqrt{5}$ is an integer or 5 is irrational is :

(1) $\sqrt{5}$ is irrational or 5 is an integer.

(2) $\sqrt{5}$ is not an integer and 5 is not irrational.

(3) $\sqrt{5}$ is an integer and 5 is irrational.

(4) $\sqrt{\ }$ is not an integer or 5 is not irrational.

Sol. $p = \sqrt{5}$ is an integer.

q : 5 is irrational

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

$= \sqrt{5}$ is not an integer and 5 is not irrational.

10. Let the observations $x_i (1 \le i \le 10)$ satisfy the equations, $\sum_{i=1}^{10}(x_i - 5) = 10$ and $\sum_{i=1}^{10}(x_i - 5)^2 = 40$.

If μ and λ are the mean and the variance of the observations, $x_1 - 3, x_2 - 3, ..., x_{10} - 3$, then the ordered pair (μ, λ) is equal to :

(1) (6, 6) (2) (3, 6)

(3) (6, 3) (4) (3, 3)

NTA Ans. (4)

Sol. $\sum_{i=1}^{10}(x_i - 5) = 10$

$\Rightarrow$ Mean of observation $x_i - 5 = \frac{1}{10}\sum_{i=1}^{3}(x_i - 5) = 1$

$\Rightarrow$ μ = mean of observation $(x_i - 3)$

= (mean of observation $(x_i - 5)$) + 2

= 1 + 2 = 3

Variance of observation

$x_i - 5 = \frac{1}{10}\sum_{i=1}^{10}(x_i - 5)^2 - (\text{Mean of } (x_i - 5))^2 = 3$

$\Rightarrow$ λ = variance of observation $(x_i - 3)$

= variance of observation $(x_i - 5) = 3$

∴ (μ, λ) = (3, 3)

11. The product $2^{\frac{1}{4}} \cdot 4^{\frac{1}{16}} \cdot 8^{\frac{1}{48}} \cdot 16^{\frac{1}{128}} \cdot ....$ to ∞ is equal to :

(1) $2^{\frac{1}{2}}$ (2) $2^{\frac{1}{4}}$ (3) 2 (4) 1

NTA Ans. (1)

Sol. $2^{\frac{1}{4}} \cdot 4^{\frac{1}{16}} \cdot 8^{\frac{1}{48}} \cdot 16^{\frac{1}{128}} \cdot .... \infty$

$= 2^{\frac{1}{4}} \cdot 2^{\frac{2}{16}} \cdot 2^{\frac{3}{48}} \cdot 2^{\frac{4}{128}} \cdot ....\infty$

$= 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdot 2^{\frac{1}{16}} \cdot 2^{\frac{1}{32}} \cdot ....\infty$

12. A circle touches the y-axis at the point (0, 4) and passes through the point (2, 0). Which of the following lines is not a tangent to this circle ?

(1) 3x – 4y – 24 = 0 (2) 3x + 4y – 6 = 0

(3) 4x + 3y – 8 = 0 (4) 4x – 3y + 17 = 0

NTA Ans. (3)

Sol. Equation of family of circle touching y-axis at (0, 4) is given by $(x - 0)^2 + (y - 4)^2 + \lambda x = 0$.

∵ It passes through (2, 0)

$\Rightarrow$ λ = –10.

$\Rightarrow$ Required circle is $(x - 0)^2 + (y - 4)^2 - 10x = 0$

$\Rightarrow$ $x^2 + y^2 - 10x - 8y + 16 = 0$

center of circle ≡ (5, 4) and radius = 5

distance of 4x + 3y – 8 = 0 from (5, 4)

$= \left|\frac{24}{5}\right| \ne \text{radius}$

13. If e_1 and e_2 are the eccentricities of the ellipse, $\frac{x^2}{18} + \frac{y^2}{4} = 1$ and the hyperbola, $\frac{x^2}{9} - \frac{y^2}{4} = 1$ respectively and (e_1, e_2) is a point on the ellipse, $15x^2 + 3y^2 = k$, then k is equal to :

(1) 15 (2) 14 (3) 17 (4) 16

NTA Ans. (4)

Sol. $e_1 = \sqrt{1 - \frac{4}{18}} = \frac{\sqrt{7}}{3}$

$e_2 = \sqrt{1 + \frac{4}{9}} = \frac{\sqrt{13}}{3}$

∵ (e_1, e_2) lies on $15x^2 + 3y^2 = k$

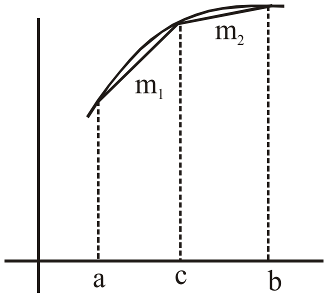
$\Rightarrow$ $15e_1^2 + 3e_2^2 = k$

$\Rightarrow$ k = 16

14. Let *f* be any function continuous on [a, b] and twice differentiable on (a, b). If for all x ∈ (a, b), $f'(x) > 0$ and $f''(x) < 0$, then for any c ∈ (a, b), $\frac{f(c) - f(a)}{f(b) - f(c)}$ is greater than :

(1) $\frac{b+a}{b-a}$ (2) $\frac{b-c}{c-a}$ (3) $\frac{c-a}{b-c}$ (4) 1

Sol.



it is clear from graph that $m_1 > m_2$

$$\Rightarrow \quad \frac{f(c)-f(a)}{c-a} > \frac{f(b)-f(c)}{b-c}$$

$$\Rightarrow \quad \frac{f(c)-f(a)}{f(b)-f(c)} > \frac{c-a}{b-c}$$

15. If for some α and β in R, the intersection of the following three places

$x + 4y - 2z = 1$

$x + 7y - 5z = \beta$

$x + 5y + \alpha z = 5$

is a line in R^3, then $\alpha + \beta$ is equal to :

(1) 10 (2) –10 (3) 2 (4) 0

NTA Ans. (1)

Sol. For planes to intersect on a line

$\Rightarrow$ there should be infinite solution of the given system of equations

for infinite solutions

$$\Delta = \begin{vmatrix} 1 & 4 & -2 \\ 1 & 7 & -5 \\ 1 & 5 & \alpha \end{vmatrix} = 0 \Rightarrow \quad 3\alpha + 9 = 0 \Rightarrow \alpha = -3$$

$$\Delta_z = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 7 & \beta \\ 1 & 5 & 5 \end{vmatrix} = 0 \Rightarrow 13 - \beta = 0 \Rightarrow \beta = 13$$

Also for $\alpha = -3$ and $b = 13$ $\Delta_x = \Delta_y = 0$

$\therefore \quad \alpha + \beta = -3 + 13 = 10$

16. The integral $\int \frac{dx}{(x+4)^{\frac{8}{7}}(x-3)^{\frac{6}{7}}}$ is equal to :

(where C is a constant of integration)

(1) $\left(\frac{x-3}{x+4}\right)^{\frac{1}{7}} + C$ (2) $-\left(\frac{x-3}{x+4}\right)^{-\frac{1}{7}} + C$

(3) $\frac{1}{2}\left(\frac{x-3}{x+4}\right)^{\frac{3}{7}} + C$ (4) $-\frac{1}{13}\left(\frac{x-3}{x+4}\right)^{-\frac{13}{7}} + C$

Sol. $I = \int \frac{dx}{(x+4)^{\frac{8}{7}}(x-3)^{\frac{6}{7}}} = \int \frac{dx}{\left(\frac{x+4}{x-3}\right)^{\frac{8}{7}}(x-3)^2}$

Let $\frac{x+4}{x-3} = t \Rightarrow \frac{dx}{(x-3)^2} = -\frac{1}{7}dt$

$$\Rightarrow \quad I = -\frac{1}{7}\int \frac{dt}{t^{8/7}} = -\frac{1}{7}\int t^{-8/7}dt$$

$$= t^{-1/7} + C = +\left(\frac{x+4}{x-3}\right)^{-1/7} + C = \left(\frac{x-3}{x+4}\right)^{1/7} + C$$

17. Let C be the centroid of the triangle with vertices (3, –1), (1, 3) and (2, 4). Let P be the point of intersection of the lines $x + 3y - 1 = 0$ and $3x - y + 1 = 0$. Then the line passing through the points C and P also passes through the point :

(1) (7, 6) (2) (–9, –6)

(3) (–9, –7) (4) (9, 7)

NTA Ans. (2)

Sol. Centroid of $\Delta = (2, 2)$

line passing through intersection of

$x + 3y - 1 = 0$ and

$3x - y + 1 = 0$, be given by

$(x + 3y - 1) + \lambda(3x - y + 1) = 0$

$\because$ It passes through (2, 2)

$\Rightarrow \quad 7 + 5\lambda = 0 \Rightarrow \lambda = -\frac{7}{5}$

$\therefore$ Required line is $8x - 11y + 6 = 0$

$\because$ (–9, –6) satisfies this equation.

18. If $f(x) = \begin{cases} \dfrac{\sin(a+2)x + \sin x}{x} & ; \; x < 0 \\ b & ; \; x = 0 \\ \dfrac{(x+3x^2)^{\frac{1}{3}} - x^{-\frac{1}{3}}}{x^{\frac{4}{3}}} & ; \; x > 0 \end{cases}$

is continuous at $x = 0$, then $a + 2b$ is equal to :

(1) –1 (2) 1 (3) –2 (4) 0

Sol. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left(\frac{\sin(a+2)x}{x} + \frac{\sin x}{x}\right) = a+3$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\frac{(x+3x^2)^{1/3} - x^{1/3}}{x^{4/3}}$

$= \lim_{x\to 0}\frac{(1+3x)^{1/3}-1}{x} = 1$

$f(0) = b$

for continuity at $x = 0$

$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$

$\Rightarrow\ a + 3 = b = 1$

$\therefore\ a = -2,\ b = 1$

$\therefore\ a + 2b = 0$

19. The value of

$$\cos^3\left(\frac{\pi}{8}\right)\cdot\cos\left(\frac{3\pi}{8}\right) + \sin^3\left(\frac{\pi}{8}\right)\cdot\sin\left(\frac{3\pi}{8}\right)$$

is :

(1) $\frac{1}{4}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\frac{1}{2\sqrt{2}}$ (4) $\frac{1}{2}$

NTA Ans. (3)

Sol. $\cos^3\frac{\pi}{8}\cdot\sin\frac{\pi}{8} + \sin^3\frac{\pi}{8}\cdot\cos\frac{\pi}{8}$

$= \sin\frac{\pi}{8}.\cos\frac{\pi}{8} = \frac{1}{2}\sin\frac{\pi}{4} = \frac{1}{2\sqrt{2}}$

20. If for all real triplets (a, b, c), $f(x) = a + bx + cx^2$; then $\int_0^1 f(x)dx$ is equal to :

(1) $\frac{1}{2}\left\{f(1) + 3f\left(\frac{1}{2}\right)\right\}$

(2) $2\left\{3f(1) + 2f\left(\frac{1}{2}\right)\right\}$

(3) $\frac{1}{6}\left\{f(0) + f(1) + 4f\left(\frac{1}{2}\right)\right\}$

(4) $\frac{1}{3}\left\{f(0) + f\left(\frac{1}{2}\right)\right\}$

NTA Ans. (3)

Sol. $f(x) = a + bx + cx^2$

$\int_0^1 f(x)dx = \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_0^1$

$= a + \frac{b}{2} + \frac{c}{3} = \frac{1}{6}[6a + 3b + c]$

$= \frac{1}{6}\left[f(0) + f(1) + 4f\left(\frac{1}{2}\right)\right]$

21. The coefficient of x^4 is the expansion of $(1 + x + x^2)^{10}$ is ———.

NTA Ans. (615.00)

Sol. $(1 + x + x^2)^{10}$

$= {}^{10}C_0 + {}^{10}C_1 x(1 + x) + {}^{10}C_2 x^2(1 + x)^2$

$+ {}^{10}C_3 x^3(1 + x)^3 + {}^{10}C_4 x^4(1 + x)^4 + ...$

Coeff. of $x^4 = {}^{10}C_2 + {}^{10}C_3 \times {}^3C_1 + {}^{10}C_4 = 615$.

22. The number of distinct solutions of the equation $\log_{\frac{1}{2}}|\sin x| = 2 - \log_{\frac{1}{2}}|\cos x|$ in the interval $[0, 2\pi]$, is ———.

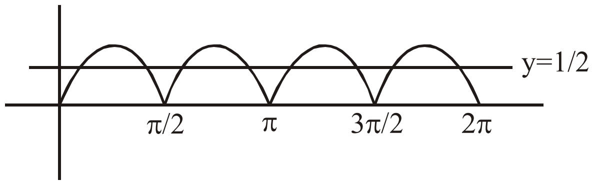
NTA Ans. (8.00)

Sol. $\log_{1/2}|\sin x| = 2 - \log_{1/2}|\cos x|;\ x \in [0, 2\pi]$

$\Rightarrow\ \log_{1/2}|\sin x| + \log_{1/2}|\cos x| = 2$

$\Rightarrow\ \log_{1/2}(|\sin x\cos x|) = 2$

$\Rightarrow\ |\sin x\cos x| = \frac{1}{4}\ \Rightarrow\ |\sin 2x| = \frac{1}{2}$



$\Rightarrow$ 8 solutions

23. If for $x \ge 0$, $y = y(x)$ is the solution of the differential equation

$(x + 1)dy = ((x + 1)^2 + y - 3)dx$, $y(2) = 0$,

then $y(3)$ is equal to ———.

NTA Ans. (3.00)

Sol. $(x + 1)dy - ydx = ((x + 1)^2 - 3)dx$

$$\Rightarrow \frac{(x+1)dy - ydx}{(x+1)^2} = \left(1 - \frac{3}{(x+1)^2}\right)dx$$

$$\Rightarrow d\left(\frac{y}{(x+1)}\right) = \left(1 - \frac{3}{(x+1)^2}\right)dx$$

integrating both sides

$$\frac{y}{x+1} = x + \frac{3}{(x+1)} + C$$

Given $y(2) = 0 \quad \Rightarrow \quad c = -3$

$$\therefore \quad y = (x+1)\left(x + \frac{3}{(x+1)} - 3\right)$$

$$\therefore \quad y(3) = 3.00$$

24. If the vectors, $\vec{p} = (a+1)\hat{i} + a\hat{j} + a\hat{k}$,

$\vec{q} = a\hat{i} + (a+1)\hat{j} + a\hat{k}$ and

$\vec{r} = a\hat{i} + a\hat{j} + (a+1)\hat{k}$ $(a \in R)$ are coplanar and $3(\vec{p}.\vec{q})^2 - \lambda|\vec{r} \times \vec{q}|^2 = 0$, then the value of λ is ______.

NTA Ans. (1.00)

Sol. $\vec{p} = (a+1)\hat{i} + a\hat{j} + a\hat{k}$,

$\vec{q} = a\hat{i} + (a+1)\hat{j} + a\hat{k}$ and

$\vec{r} = a\hat{i} + a\hat{j} + (a+1)\hat{k}$

$\because \quad \vec{p}, \vec{q}, \vec{r}$ are coplanar

$$\Rightarrow \quad [\vec{p} \;\; \vec{q} \;\; \vec{r}] = 0$$

$$\Rightarrow \begin{vmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 3a + 1 = 0 \quad \Rightarrow \quad a = -\frac{1}{3}$$

$$\vec{p}.\vec{q} = -\frac{1}{3}, \quad \vec{r}.\vec{q} = -\frac{1}{3}$$

$$|\vec{r}|^2 = |\vec{q}|^2 = \frac{2}{3}$$

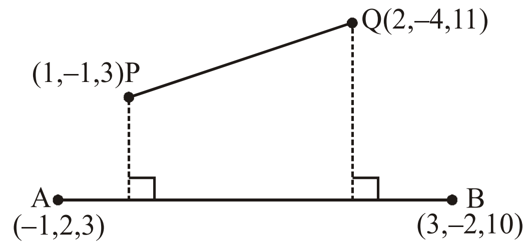
$$\therefore \quad 3(\vec{p}.\vec{q})^2 - \lambda|\vec{r} \times \vec{q}|^2 = 0$$

$$\Rightarrow \quad \lambda = \frac{3(\vec{p}.\vec{q})^2}{|\vec{r} \times \vec{q}|^2} = \frac{3(\vec{p}.\vec{q})^2}{|\vec{r}|^2|\vec{q}|^2 - (\vec{r}.\vec{q})^2} = 1.00$$

25. The projection of the line segment joining the points (1, –1, 3) and (2, –4, 11) on the line joining the points (–1, 2, 3) and (3, –2, 10) is ______.

NTA Ans. (8.00)

Sol.



$$\text{Projection of } \overrightarrow{PQ} \text{ on } \overrightarrow{AB} = \left|\frac{\overrightarrow{PQ} \cdot \overrightarrow{AB}}{|\overrightarrow{AB}|}\right|$$

$$= \left|\frac{(\hat{i} - 3\hat{j} + 8\hat{k}) \cdot (4\hat{i} - 4\hat{j} + 7\hat{k})}{9}\right| = 8$$