

## FINAL JEE–MAIN EXAMINATION – MARCH, 2021

(Held On Tuesday 16<sup>th</sup> March, 2021) TIME : 3 : 00 PM to 6 : 00 PM

### MATHEMATICS

### TEST PAPER WITH SOLUTION

#### SECTION-A

1. The maximum value of

$$f(x) = \begin{vmatrix} \sin^2 x & 1 + \cos^2 x & \cos 2x \\ 1 + \sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & \sin 2x \end{vmatrix}, x \in \mathbb{R} \text{ is:}$$

- (1)  $\sqrt{7}$       (2)  $\frac{3}{4}$       (3)  $\sqrt{5}$       (4) 5

**Official Ans by NTA (3)**

**Sol.**  $C_1 + C_2 \rightarrow C_1$

$$\begin{vmatrix} 2 & 1 + \cos^2 x & \cos 2x \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

$$R_1 - R_2 \rightarrow R_1$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 2 & \cos^2 x & \cos 2x \\ 1 & \cos^2 x & \sin 2x \end{vmatrix}$$

Open w.r.t.  $R_1$

$$-(2 \sin 2x - \cos 2x)$$

$$\cos 2x - 2 \sin 2x = f(x)$$

$$f(x)|_{\max} = \sqrt{1+4} = \sqrt{5}$$

2. Let A denote the event that a 6-digit integer formed by 0, 1, 2, 3, 4, 5, 6 without repetitions, be divisible by 3. Then probability of event A is equal to :

- (1)  $\frac{9}{56}$       (2)  $\frac{4}{9}$       (3)  $\frac{3}{7}$       (4)  $\frac{11}{27}$

**Official Ans by NTA (2)**

**Sol.** Total cases :

$$\underline{6} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2}$$

$$n(s) = 6 \cdot 6!$$

Favourable cases :

Number divisible by 3  $\equiv$

Sum of digits must be divisible by 3

**Case-I**

1, 2, 3, 4, 5, 6

Number of ways = 6!

**Case-II**

0, 1, 2, 4, 5, 6

Number of ways = 5·5!

**Case-III**

0, 1, 2, 3, 4, 5

Number of ways = 5·5!

n(favourable) = 6! + 2·5·5!

$$P = \frac{6! + 2 \cdot 5 \cdot 5!}{6 \cdot 6!} = \frac{4}{9}$$

3. Let  $\alpha \in \mathbb{R}$  be such that the function

$$f(x) = \begin{cases} \frac{\cos^{-1}(1-\{x\}^2) \sin^{-1}(1-\{x\})}{\{x\} - \{x\}^3}, & x \neq 0 \\ \alpha, & x = 0 \end{cases}$$

is continuous at  $x = 0$ , where  $\{x\} = x - [x]$ ,  $[x]$  is the greatest integer less than or equal to  $x$ .

Then :

(1)  $\alpha = \frac{\pi}{\sqrt{2}}$       (2)  $\alpha = 0$

(3) no such  $\alpha$  exists      (4)  $\alpha = \frac{\pi}{4}$

**Official Ans by NTA (3)**

**Sol.**

$$\lim_{x \rightarrow 0^+} f(x) = f(0) = \lim_{x \rightarrow 0^-} f(x)$$

$$\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(1-x^2) \cdot \sin^{-1}(1-x)}{x(1-x)(1+x)}$$

$$\lim_{x \rightarrow 0^+} \frac{\cos^{-1}(1-x^2)}{x} \cdot \frac{\pi}{4}$$

Let  $1 - x^2 = \cos \theta$

$$\frac{\pi}{2} \lim_{x \rightarrow 0^+} \frac{\theta}{\sqrt{1 - \cos \theta}}$$

$$\frac{\pi}{2} \lim_{\theta \rightarrow 0^+} \frac{\theta}{\sqrt{2} \sin \frac{\theta}{2}} = \frac{\pi}{\sqrt{2}}$$

Now,  $\lim_{x \rightarrow 0^-} \frac{\cos^{-1}(1 - (1+x)^2) \sin^{-1}(-x)}{(1+x) - (1+x)^3}$

$$\lim_{x \rightarrow 0^-} \frac{\frac{\pi}{2} (-\sin^{-1} x)}{(1+x)(2+x)(-x)}$$

$$\lim_{x \rightarrow 0^-} \frac{\frac{\pi}{2} \cdot \frac{\sin^{-1} x}{x}}{1 \cdot 2} = \frac{\pi}{4}$$

$\Rightarrow$  RHL  $\neq$  LHL

Function can't be continuous

$\Rightarrow$  No value of  $\alpha$  exist

4. If  $(x, y, z)$  be an arbitrary point lying on a plane P which passes through the point  $(42, 0, 0)$ ,  $(0, 42, 0)$  and  $(0, 0, 42)$ , then the value of expression

$$3 + \frac{x-11}{(y-19)^2(z-12)^2} + \frac{y-19}{(x-11)^2(z-12)^2} + \frac{z-12}{(x-11)^2(y-19)^2} - \frac{x+y+z}{14(x-11)(y-19)(z-12)}$$

(1) 0      (2) 3      (3) 39      (4) -45

**Official Ans by NTA (2)**

- Sol.** Plane passing through  $(42, 0, 0)$ ,  $(0, 42, 0)$ ,  $(0, 0, 42)$

From intercept form, equation of plane is

$$x + y + z = 42$$

$$\Rightarrow (x - 11) + (y - 19) + (z - 12) = 0$$

let  $a = x - 11$ ,  $b = y - 19$ ,  $c = z - 12$

$$a + b + c = 0$$

$$3 + \frac{a}{b^2c^2} + \frac{b}{a^2c^2} + \frac{c}{a^2b^2} - \frac{42}{14abc}$$

$$3 + \frac{a^3 + b^3 + c^3 - 3abc}{a^2b^2c^2}$$

If  $a + b + c = 0$

$$\Rightarrow a^3 + b^3 + c^3 = 3abc$$

$$\Rightarrow 3$$

5. Consider the integral

$$I = \int_0^{10} \frac{[x] e^{[x]}}{e^{x-1}} dx,$$

where  $[x]$  denotes the greatest integer less than or equal to  $x$ . Then the value of  $I$  is equal to:

- (1)  $9(e - 1)$       (2)  $45(e + 1)$   
 (3)  $45(e - 1)$       (4)  $9(e + 1)$

**Official Ans by NTA (3)**

**Sol.**  $I = \int_0^{10} [x] \cdot e^{[x]-x+1} dx$

$$I = \int_0^1 0 dx + \int_1^2 1 \cdot e^{2-x} dx + \int_2^3 2 \cdot e^{3-x} dx + \dots + \int_9^{10} 9 \cdot e^{10-x} dx$$

$$\Rightarrow I = \sum_{n=0}^9 \int_n^{n+1} n \cdot e^{n+1-x} dx$$

$$= - \sum_{n=0}^9 n \left( e^{n+1-x} \right)_n^{n+1}$$

$$= - \sum_{n=0}^9 n \cdot (e^0 - e^1)$$

$$= (e-1) \sum_{n=0}^9 n$$

$$= (e-1) \cdot \frac{9 \cdot 10}{2}$$

6. Let C be the locus of the mirror image of a point on the parabola  $y^2 = 4x$  with respect to the line  $y = x$ . Then the equation of tangent to C at P(2,1) is :

- (1)  $x - y = 1$                       (2)  $2x + y = 5$   
 (3)  $x + 3y = 5$                     (4)  $x + 2y = 4$

**Official Ans by NTA (1)**

**Sol.** Given  $y^2 = 4x$

Mirror image on  $y = x \Rightarrow C : x^2 = 4y$

$$2x = 4 \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

$$\left. \frac{dy}{dx} \right|_{P(2,1)} = \frac{2}{2} = 1$$

Equation of tangent at (2, 1)

$$\Rightarrow y - 1 = 1(x - 2)$$

$$\Rightarrow x - y = 1$$

7. If  $y = y(x)$  is the solution of the differential equation  $\frac{dy}{dx} + (\tan x)y = \sin x$ ,  $0 \leq x \leq \frac{\pi}{3}$ , with

$y(0) = 0$ , then  $y\left(\frac{\pi}{4}\right)$  equal to :

(1)  $\frac{1}{4} \log_e 2$                       (2)  $\left(\frac{1}{2\sqrt{2}}\right) \log_e 2$

(3)  $\log_e 2$                       (4)  $\frac{1}{2} \log_e 2$

**Official Ans by NTA (2)**

**Sol.**  $\frac{dy}{dx} + (\tan x)y = \sin x$ ;  $0 \leq x \leq \frac{\pi}{3}$

$$\text{I.F.} = e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$$

$$y \sec x = \int \tan x \, dx$$

$$y \sec x = \int \tan x \, dx$$

$$y \sec x = \ln |\sec x| + C$$

$$x = 0, y = 0 \Rightarrow \therefore c = 0$$

$$y \sec x = \ln |\sec x|$$

$$y = \cos x \cdot \ln |\sec x|$$

$$y \Big|_{x=\frac{\pi}{4}} = \left(\frac{1}{\sqrt{2}}\right) \cdot \ln \sqrt{2}$$

$$y \Big|_{x=\frac{\pi}{4}} = \frac{1}{2\sqrt{2}} \log_e 2$$

8. Let  $A = \{2, 3, 4, 5, \dots, 30\}$  and ' $\simeq$ ' be an equivalence relation on  $A \times A$ , defined by  $(a, b) \simeq (c, d)$ , if and only if  $ad = bc$ . Then the number of ordered pairs which satisfy this equivalence relation with ordered pair (4, 3) is equal to :

- (1) 5                      (2) 6                      (3) 8                      (4) 7

**Official Ans by NTA (4)**

**Sol.**  $A = \{2, 3, 4, 5, \dots, 30\}$

$$(a, b) \simeq (c, d) \Rightarrow ad = bc$$

$$(4, 3) \simeq (c, d) \Rightarrow 4d = 3c$$

$$\Rightarrow \frac{4}{3} = \frac{c}{d}$$

$$\frac{c}{d} = \frac{4}{3} \quad \& \quad c, d \in \{2, 3, \dots, 30\}$$

$$(c, d) = \{(4, 3), (8, 6), (12, 9), (16, 12), (20, 15), (24, 18), (28, 21)\}$$

No. of ordered pair = 7

9. Let the lengths of intercepts on x-axis and y-axis made by the circle  $x^2 + y^2 + ax + 2ay + c = 0$ , ( $a < 0$ ) be  $2\sqrt{2}$  and  $2\sqrt{5}$ , respectively. Then the shortest distance from origin to a tangent to this circle which is perpendicular to the line  $x + 2y = 0$ , is equal to :

- (1)  $\sqrt{11}$                       (2)  $\sqrt{7}$                       (3)  $\sqrt{6}$                       (4)  $\sqrt{10}$

**Official Ans by NTA (3)**

**Sol.**  $x^2 + y^2 + ax + 2ay + c = 0$

$$2\sqrt{g^2 - c} = 2\sqrt{\frac{a^2}{4} - c} = 2\sqrt{2}$$

$$\Rightarrow \frac{a^2}{4} - c = 2 \quad \dots(1)$$

$$\sqrt{2 - c} = \sqrt{2 - c} = \sqrt{2}$$

$$\Rightarrow a^2 - c = 5 \quad \dots(2)$$

(1) & (2)

$$\frac{3a^2}{4} = 3 \Rightarrow a = -2 \quad (a < 0)$$

$$\therefore c = -1$$

$$\text{Circle} \Rightarrow x^2 + y^2 - 2x - 4y - 1 = 0$$

$$\Rightarrow (x - 1)^2 + (y - 2)^2 = 6$$

$$\text{Given } x + 2y = 0 \Rightarrow m = -\frac{1}{2}$$

$$m_{\text{tangent}} = 2$$

Equation of tangent

$$\Rightarrow (y - 2) = 2(x - 1) \pm \sqrt{6}\sqrt{1+4}$$

$$\Rightarrow 2x - y \pm \sqrt{30} = 0$$

$$\text{Perpendicular distance from } (0, 0) = \left| \frac{\pm\sqrt{30}}{\sqrt{4+1}} \right| = \sqrt{6}$$

10. The least value of  $|z|$  where  $z$  is complex number which satisfies the inequality

$$\exp\left(\frac{(|z|+3)(|z|-1)}{||z|+1|} \log_e 2\right) \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|,$$

$i = \sqrt{-1}$ , is equal to :

- (1) 3      (2)  $\sqrt{5}$       (3) 2      (4) 8

Official Ans by NTA (1)

Sol. 
$$\exp\left(\frac{(|z|+3)(|z|-1)}{||z|+1|} \ln 2\right) \geq \log_{\sqrt{2}} |5\sqrt{7} + 9i|$$

$$\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \geq \log_{\sqrt{2}}(16)$$

$$\Rightarrow 2^{\frac{(|z|+3)(|z|-1)}{(|z|+1)}} \geq 2^3 \Rightarrow \frac{(|z|+3)(|z|-1)}{(|z|+1)} \geq 3$$

$$\Rightarrow (|z| + 3)(|z| - 1) \geq 3(|z| + 1)$$

$$|z|^2 + 2|z| - 3 \geq 3|z| + 3$$

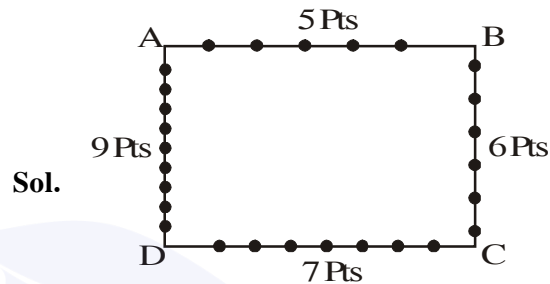
$$\Rightarrow |z|^2 + |z| - 6 \geq 0$$

$$\Rightarrow (|z| - 3)(|z| + 2) \geq 0 \Rightarrow |z| - 3 \geq 0$$

$$\Rightarrow |z| \geq 3 \Rightarrow |z|_{\min} = 3$$

11. Consider a rectangle ABCD having 5, 7, 6, 9 points in the interior of the line segments AB, CD, BC, DA respectively. Let  $\alpha$  be the number of triangles having these points from different sides as vertices and  $\beta$  be the number of quadrilaterals having these points from different sides as vertices. Then  $(\beta - \alpha)$  is equal to :
- (1) 795      (2) 1173      (3) 1890      (4) 717

Official Ans by NTA (4)



Sol.

$\alpha$  = Number of triangles

$$\begin{aligned} \alpha &= 5 \cdot 6 \cdot 7 + 5 \cdot 7 \cdot 9 + 5 \cdot 6 \cdot 9 + 6 \cdot 7 \cdot 9 \\ &= 210 + 315 + 270 + 378 \\ &= 1173 \end{aligned}$$

$\beta$  = Number of Quadrilateral

$$\beta = 5 \cdot 6 \cdot 7 \cdot 9 = 1890$$

$$\beta - \alpha = 1890 - 1173 = 717$$

12. If the point of intersections of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1 \text{ and the circle } x^2 + y^2 = 4b, \quad b > 4$$

lie on the curve  $y^2 = 3x^2$ , then  $b$  is equal to:

- (1) 12      (2) 5      (3) 6      (4) 10

Official Ans by NTA (1)

Sol.

$$y^2 = 3x^2$$

$$\text{and } x^2 + y^2 = 4b$$

Solve both we get

$$\text{so } x^2 = b$$

$$\frac{x^2}{16} + \frac{3x^2}{b^2} = 1$$

$$\frac{b}{16} + \frac{3}{b} = 1$$

$$b^2 - 16b + 48 = 0$$

$$(b - 12)(b - 4) = 0$$

13. Given that the inverse trigonometric functions take principal values only. Then, the number of real values of  $x$  which satisfy

$$\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1} x \text{ is equal to:}$$

- (1) 2      (2) 1      (3) 3      (4) 0

**Official Ans by NTA (3)**

**Sol.**  $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1} x$

$$\sin^{-1}\left(\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right) = \sin^{-1} x$$

$$\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}} + \frac{4x}{5}\sqrt{1-\frac{9x^2}{25}} = x$$

$$x = 0, 3\sqrt{25-16x^2} + 4\sqrt{25-9x^2} = 25$$

$$4\sqrt{25-9x^2} = 25 - 3\sqrt{25-16x^2} \text{ squaring we get}$$

$$16(25-9x^2) = 625 + 9(25-16x^2) - 150\sqrt{25-16x^2}$$

$$400 = 625 + 225 - 150\sqrt{25-16x^2}$$

$$\sqrt{25-16x^2} = 3 \Rightarrow 25 - 16x^2 = 9$$

$$\Rightarrow x^2 = 1$$

Put  $x = 0, 1, -1$  in the original equation

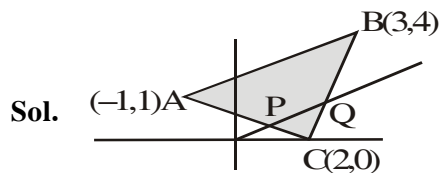
We see that all values satisfy the original equation.

Number of solution = 3

14. Let  $A(-1, 1)$ ,  $B(3, 4)$  and  $C(2, 0)$  be given three points. A line  $y = mx$ ,  $m > 0$ , intersects lines  $AC$  and  $BC$  at point  $P$  and  $Q$  respectively. Let  $A_1$  and  $A_2$  be the areas of  $\triangle ABC$  and  $\triangle PQC$  respectively, such that  $A_1 = 3A_2$ , then the value of  $m$  is equal to :

- (1)  $\frac{4}{15}$       (2) 1      (3) 2      (4) 3

**Official Ans by NTA (2)**



**Sol.**

$$P \equiv (x_1, mx_1)$$

$$Q \equiv (x_2, mx_2)$$

$$A_1 = \frac{1}{2} \begin{vmatrix} 3 & 4 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} = \frac{13}{2}$$

$$A_2 = \frac{1}{2} \begin{vmatrix} x_1 & mx_1 & 1 \\ x_2 & mx_2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$A_2 = \frac{1}{2} |2(mx_1 - mx_2)| = m|x_1 - x_2|$$

$$A_1 = 3A_2 \Rightarrow \frac{13}{2} = 3m|x_1 - x_2|$$

$$\Rightarrow |x_1 - x_2| = \frac{16}{6m}$$

$$AC : x + 3y = 2$$

$$BC : y = 4x - 8$$

$$P : x + 3y = 2 \text{ \& } y = mx \Rightarrow x_1 = \frac{2}{1+3m}$$

$$Q : y = 4x - 8 \text{ \& } y = mx \Rightarrow x_2 = \frac{8}{4-m}$$

$$|x_1 - x_2| = \left| \frac{2}{1+3m} - \frac{8}{4-m} \right|$$

$$= \left| \frac{-26m}{(1+3m)(4-m)} \right| = \frac{26m}{(3m+1)|m-4|}$$

$$= \frac{26m}{(3m+1)(4-m)}$$

$$|x_1 - x_2| = \frac{13}{6m}$$

$$\frac{26m}{(3m+1)(4-m)} = \frac{13}{6m}$$

$$\Rightarrow 12m^2 = -(3m+1)(m-4)$$

$$\Rightarrow 12m^2 = -(3m^2 - 11m - 4)$$

$$\Rightarrow 15m^2 - 11m - 4 = 0$$

$$\Rightarrow 15m^2 - 15m + 4m - 4 = 0$$

$$\Rightarrow (15m+4)(m-1) = 0$$

$$\Rightarrow m = 1$$

15. Let  $f$  be a real valued function, defined on  $\mathbb{R} - \{-1, 1\}$  and given by

$$f(x) = 3 \log_e \left| \frac{x-1}{x+1} \right| - \frac{2}{x-1}$$

Then in which of the following intervals, function  $f(x)$  is increasing?

(1)  $(-\infty, -1) \cup \left( \left[ \frac{1}{2}, \infty \right) - \{1\} \right)$

(2)  $(-\infty, \infty) - \{-1, 1\}$

(3)  $\left( -1, \frac{1}{2} \right]$

(4)  $\left( -\infty, \frac{1}{2} \right] - \{-1\}$

**Official Ans by NTA (1)**

**Sol.**  $f(x) = 3 \ln(x-1) - 3 \ln(x+1) - \frac{2}{x-1}$

$$f'(x) = \frac{3}{x-1} - \frac{3}{x+1} + \frac{2}{(x-1)^2}$$

$$f'(x) = \frac{4(2x-1)}{(x-1)^2(x+1)}$$

$$f'(x) \geq 0$$

$$\Rightarrow x \in (-\infty, -1) \cup \left[ \frac{1}{2}, 1 \right) \cup (1, \infty)$$

16. Let  $f : S \rightarrow S$  where  $S = (0, \infty)$  be a twice differentiable function such that  $f(x+1) = xf(x)$ . If  $g : S \rightarrow \mathbb{R}$  be defined as  $g(x) = \log_e f(x)$ , then the value of  $|g''(5) - g''(1)|$  is equal to :

(1)  $\frac{205}{144}$  (2)  $\frac{197}{144}$  (3)  $\frac{187}{144}$  (4) 1

**Official Ans by NTA (1)**

**Sol.**  $\ln f(x+1) = \ln(xf(x))$

$$\ln f(x+1) = \ln x + \ln f(x)$$

$$\Rightarrow g(x+1) = \ln x + g(x)$$

$$\Rightarrow g(x+1) - g(x) = \ln x$$

$$\Rightarrow g''(x+1) - g''(x) = -\frac{1}{x^2}$$

Put  $x = 1, 2, 3, 4$

$$g''(2) - g''(1) = -\frac{1}{1^2} \quad \dots(1)$$

$$g''(3) - g''(2) = -\frac{1}{2^2} \quad \dots(2)$$

$$g''(4) - g''(3) = -\frac{1}{3^2} \quad \dots(3)$$

$$g''(5) - g''(4) = -\frac{1}{4^2} \quad \dots(4)$$

Add all the equation we get

$$g''(5) - g''(1) = -\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}$$

$$|g''(5) - g''(1)| = \frac{205}{144}$$

17. Let  $P(x) = x^2 + bx + c$  be a quadratic polynomial

with real coefficients such that  $\int_0^1 P(x) dx = 1$  and

$P(x)$  leaves remainder 5 when it is divided by  $(x-2)$ . Then the value of  $9(b+c)$  is equal to:

(1) 9 (2) 15 (3) 7 (4) 11

**Official Ans by NTA (3)**

**Sol.**  $\int_0^1 (x^2 + bx + c) dx = 1$

$$\frac{1}{3} + \frac{b}{2} + c = 1 \Rightarrow \frac{b}{2} + c = \frac{2}{3}$$

$$3b + 6c = 4 \quad \dots(1)$$

$$P(2) = 5$$

$$4 + 2b + c = 5$$

$$2b + c = 1 \quad \dots(2)$$

From (1) & (2)

$$b = \frac{2}{9} \quad \& \quad c = \frac{5}{9}$$

$$9(b + c) = 7$$

- 18.** If the foot of the perpendicular from point (4, 3, 8) on the line  $L_1: \frac{x-a}{l} = \frac{y-2}{3} = \frac{z-b}{4}$ ,  $l \neq 0$  is (3, 5, 7), then the shortest distance between the line  $L_1$  and line

$$L_2: \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} \text{ is equal to :}$$

$$(1) \frac{1}{2} \quad (2) \frac{1}{\sqrt{6}} \quad (3) \sqrt{\frac{2}{3}} \quad (4) \frac{1}{\sqrt{3}}$$

**Official Ans by NTA (2)**

- Sol.** (3,5,7) satisfy the line  $L_1$

$$\frac{3-a}{l} = \frac{5-2}{3} = \frac{7-b}{4}$$

$$\frac{3-a}{l} = 1 \quad \& \quad \frac{7-b}{4} = 1$$

$$a + l = 3 \quad \dots(1) \quad \& \quad b = 3 \quad \dots(2)$$

$$\vec{v}_1 = \langle 4, 3, 8 \rangle - \langle 3, 5, 7 \rangle$$

$$\vec{v}_1 = \langle 1, -2, 1 \rangle$$

$$\vec{v}_2 = \langle l, 3, 4 \rangle$$

$$\vec{v}_1 \cdot \vec{v}_2 = 0 \Rightarrow l - 6 + 4 = 0 \Rightarrow l = 2$$

$$a + l = 3 \Rightarrow a = 1$$

$$L_1: \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$L_2: \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

$$A = \langle 1, 2, 3 \rangle$$

$$B = \langle 2, 4, 5 \rangle$$

$$\vec{AB} = \langle 1, 2, 2 \rangle$$

$$\vec{p} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{q} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

$$\vec{p} \times \vec{q} = -\hat{i} + 2\hat{j} - \hat{k}$$

$$\text{Shortest distance} = \frac{|\vec{AB} \cdot (\vec{p} \times \vec{q})|}{|\vec{p} \times \vec{q}|} = \frac{1}{\sqrt{6}}$$

- 19.** Let  $C_1$  be the curve obtained by the solution of differential equation  $2xy \frac{dy}{dx} = y^2 - x^2$ ,  $x > 0$ .

Let the curve  $C_2$  be the solution of

$$\frac{2xy}{x^2 - y^2} = \frac{dy}{dx}. \text{ If both the curves pass through}$$

(1,1), then the area enclosed by the curves  $C_1$  and  $C_2$  is equal to :

$$(1) \pi - 1 \quad (2) \frac{\pi}{2} - 1 \quad (3) \pi + 1 \quad (4) \frac{\pi}{4} + 1$$

**Official Ans by NTA (2)**

**Sol.**  $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ ,  $x \in (0, \infty)$

put  $y = vx$

$$x \frac{dv}{dx} + v = \frac{v^2 - 1}{2v}$$

$$\frac{2v}{v^2 + 1} dv = -\frac{dx}{x}$$

Integrate,

$$\ln(v^2 + 1) = -\ln x + C$$

$$\ln\left(\frac{y^2}{x^2} + 1\right) = -\ln x + C$$

put  $x = 1, y = 1, C = \ln 2$

$$\ln\left(\frac{y^2}{x^2} + 1\right) = -\ln x + \ln 2$$

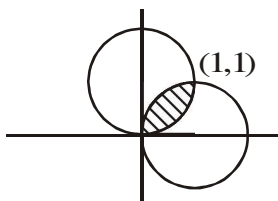
$$\Rightarrow x^2 + y^2 - 2x = 0 \quad (\text{Curve } C_1)$$

Similarly,

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

Put  $y = vx$

$$x^2 + y^2 - 2y = 0$$



$$\text{required area} = 2 \int_0^1 (\sqrt{2x - x^2} - x) dx = \frac{\pi}{2} - 1$$

20. Let  $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$  and  $\vec{b} = 2\hat{i} - 3\hat{j} + 5\hat{k}$ . If

$$\vec{r} \times \vec{a} = \vec{b} \times \vec{r}, \quad \vec{r} \cdot (\alpha \hat{i} + 2\hat{j} + \hat{k}) = 3 \quad \text{and}$$

$$\vec{r} \cdot (2\hat{i} + 5\hat{j} - \alpha \hat{k}) = -1, \quad \alpha \in \mathbb{R}, \quad \text{then the value of}$$

$\alpha + |\vec{r}|^2$  is equal to :

- (1) 9      (2) 15      (3) 13      (4) 11

**Official Ans by NTA (2)**

**Sol.**  $\vec{r} \times \vec{a} = \vec{b} \times \vec{r} \Rightarrow \vec{r} \times (\vec{a} + \vec{b}) = 0$

$$\vec{r} = \lambda(\vec{a} + \vec{b}) \Rightarrow \vec{r} = \lambda(\hat{i} + 2\hat{j} - 3\hat{k} + 2\hat{i} - 3\hat{j} + 5\hat{k})$$

$$\vec{r} = \lambda(3\hat{i} - \hat{j} + 2\hat{k}) \quad \dots(1)$$

$$\vec{r} \cdot (\alpha \hat{i} + 2\hat{j} + \hat{k}) = 3$$

Put  $\vec{r}$  from (1)  $\alpha\lambda = 1$        $\dots(2)$

$$\vec{r} \cdot (2\hat{i} + 5\hat{j} - \alpha \hat{k}) = -1$$

Put  $\vec{r}$  from (1)  $2\lambda\alpha - \lambda = 1$        $\dots(3)$

Solve (2) & (3)

$$\Rightarrow \vec{r} = 3\hat{i} - \hat{j} + 2\hat{k}$$

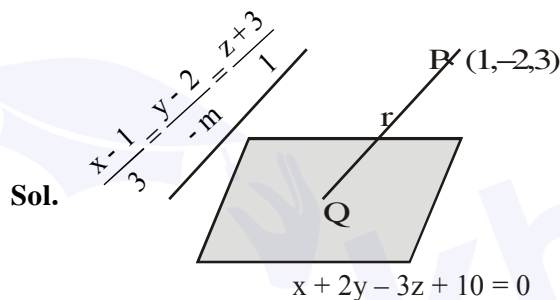
$$|\vec{r}|^2 = 14 \quad \& \quad \alpha = 1$$

$$\alpha + |\vec{r}|^2 = 15$$

**SECTION-B**

1. If the distance of the point  $(1, -2, 3)$  from the plane  $x + 2y - 3z + 10 = 0$  measured parallel to the line,  $\frac{x-1}{3} = \frac{2-y}{m} = \frac{z+3}{1}$  is  $\sqrt{\frac{7}{2}}$ , then the value of  $|m|$  is equal to \_\_\_\_\_.

**Official Ans by NTA (2)**



**Sol.**

$$\text{DC of line} = \left( \frac{3}{\sqrt{m^2 + 10}}, \frac{-m}{\sqrt{m^2 + 10}}, \frac{1}{\sqrt{m^2 + 10}} \right)$$

$$Q \equiv \left( 1 + \frac{3r}{\sqrt{m^2 + 10}}, -2 + \frac{-mr}{\sqrt{m^2 + 10}}, 3 + \frac{r}{\sqrt{m^2 + 10}} \right)$$

Q lies on  $x + 2y - 3z + 10 = 0$

$$1 + \frac{3r}{\sqrt{m^2 + 10}} - 4 - \frac{2mr}{\sqrt{m^2 + 10}} - 9 - \frac{3r}{\sqrt{m^2 + 10}} + 10 = 0$$

$$\Rightarrow \frac{r}{\sqrt{m^2 + 10}} (3 - 2m - 3) = 2$$

$$\Rightarrow \frac{r}{\sqrt{m^2 + 10}} (-2m) = 2$$

$$r^2 m^2 = m^2 + 10$$

$$\frac{7}{2} m^2 = m^2 + 10 \Rightarrow \frac{5}{2} m^2 = 10 \Rightarrow m^2 = 4$$

$$|m| = 2$$



2. Consider the statistics of two sets of observations as follows :

	Size	Mean	Variance
Observation I	10	2	2
Observation II	n	3	1

If the variance of the combined set of these two observations is  $\frac{17}{9}$ , then the value of n is equal to \_\_\_\_\_.

**Official Ans by NTA (5)**

**Sol.** 
$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1n_2}{(n_1 + n_2)^2}(\bar{x}_1 - \bar{x}_2)^2$$

$$n_1 = 10, n_2 = n, \sigma_1^2 = 2, \sigma_2^2 = 1$$

$$\bar{x}_1 = 2, \bar{x}_2 = 3, \sigma^2 = \frac{17}{9}$$

$$\frac{17}{9} = \frac{10 \times 2 + n}{n + 10} + \frac{10n}{(n + 10)^2}(3 - 2)^2$$

$$\Rightarrow \frac{17}{9} = \frac{(n + 20)(n + 10) + 10n}{(n + 10)^2}$$

$$\Rightarrow 17n^2 + 1700 + 340n = 90n + 9(n^2 + 30n + 200)$$

$$\Rightarrow 8n^2 - 20n - 100 = 0$$

$$2n^2 - 5n - 25 = 0$$

$$\Rightarrow (2n + 5)(n - 5) = 0 \Rightarrow n = \frac{-5}{2}, 5$$

↓  
(Rejected)

Hence n = 5

3. Let  $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$  and  $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  be two  $2 \times 1$  matrices with real entries such that  $A = XB$ ,

where  $X = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix}$ , and  $k \in \mathbb{R}$ . If

$$a_1^2 + a_2^2 = \frac{2}{3}(b_1^2 + b_2^2) \text{ and } (k^2 + 1)b_2^2 \neq -2b_1b_2,$$

then the value of k is \_\_\_\_\_.

**Sol.**  $A = XB$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ 1 & k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}a_1 \\ \sqrt{3}a_2 \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ b_1 + kb_2 \end{bmatrix}$$

$$b_1 - b_2 = \sqrt{3}a_1 \quad \dots(1)$$

$$b_1 + kb_2 = \sqrt{3}a_2 \quad \dots(2)$$

$$\text{Given, } a_1^2 + a_2^2 = \frac{2}{3}(b_1^2 + b_2^2)$$

$$(1)^2 + (2)^2$$

$$(b_1 + b_2)^2 + (b_1 + kb_2)^2 = 3(a_1^2 + a_2^2)$$

$$a_1^2 + a_2^2 = \frac{2}{3}b_1^2 + \frac{(1+k^2)}{3}b_2^2 + \frac{2}{3}b_1b_2(k-1)$$

$$\text{Given, } a_1^2 + a_2^2 = \frac{2}{3}b_1^2 + \frac{2}{3}b_2^2$$

On comparing we get

$$\frac{k^2 + 1}{3} = \frac{2}{3} \Rightarrow k^2 + 1 = 2$$

$$\Rightarrow k = \pm 1 \quad \dots(3)$$

$$\& \frac{2}{3}(k-1) = 0 \Rightarrow k = 1 \quad \dots(4)$$

From both we get k = 1

4. For real numbers  $\alpha, \beta, \gamma$  and  $\delta$ , if

$$\int \frac{(x^2 - 1) + \tan^{-1}\left(\frac{x^2 + 1}{x}\right)}{(x^4 + 3x^2 + 1)\tan^{-1}\left(\frac{x^2 + 1}{x}\right)} dx$$

$$= \alpha \log_e \left( \tan^{-1} \left( \frac{x^2 + 1}{x} \right) \right)$$

$$+\beta \tan^{-1}\left(\frac{\gamma(x^2-1)}{x}\right) + \delta \tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

where C is an arbitrary constant, then the value of  $10(\alpha + \beta\gamma + \delta)$  is equal to \_\_\_\_\_.

**Official Ans by NTA (6)**

**Sol.** 
$$\int \frac{(x^2-1)dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)} + \int \frac{dx}{x^4+3x^2+1}$$

$$\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(\left(x+\frac{1}{x}\right)^2+1\right)\tan^{-1}\left(x+\frac{1}{x}\right)} + \frac{1}{2} \int \frac{(x^2+1)-(x^2-1)dx}{x^4+3x^2+1}$$

Put  $\tan^{-1}\left(x+\frac{1}{x}\right) = t$

$$\int \frac{dt}{t} + \frac{1}{2} \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+5} - \frac{1}{2} \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2+1}$$

Put  $x-\frac{1}{x} = y$ ,  $x+\frac{1}{x} = z$

$$\log_e t + \frac{1}{2} \int \frac{dy}{y^2+5} - \frac{1}{2} \int \frac{dz}{z^2+1}$$

$$= \log_e \tan^{-1}\left(x+\frac{1}{x}\right) + \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right) - \frac{1}{2} \tan^{-1}\left(\frac{x^2+1}{x}\right) + C$$

$$\alpha = 1, \beta = \frac{1}{2\sqrt{5}}, \gamma = \frac{1}{\sqrt{5}}, \delta = \frac{-1}{2}$$

or

$$\alpha = 1, \beta = \frac{-1}{2\sqrt{5}}, \gamma = \frac{-1}{\sqrt{5}}, \delta = \frac{-1}{2}$$

$$10(\alpha + \beta\gamma + \delta) = 10\left(1 + \frac{1}{10} - \frac{1}{2}\right) = 6$$

**5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$f(x) = \begin{cases} x+a, & x < 0 \\ |x-1|, & x \geq 0 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x+1, & x < 0 \\ (x-1)^2+b, & x \geq 0 \end{cases}$$

where a, b are non-negative real numbers. If  $(g \circ f)(x)$  is continuous for all  $x \in \mathbb{R}$ , then  $a + b$  is equal to \_\_\_\_\_.

**Official Ans by NTA (1)**

**Sol.** 
$$g[f(x)] = \begin{cases} f(x)+1 & f(x) < 0 \\ (f(x)-1)^2+b & f(x) \geq 0 \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x+a < 0 \text{ \& } x < 0 \\ |x-1|+1 & |x-1| < 0 \text{ \& } x \geq 0 \\ (x+a-1)^2+b & x+a \geq 0 \text{ \& } x < 0 \\ (|x-1|-1)^2+b & |x-1| \geq 0 \text{ \& } x \geq 0 \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x \in (-\infty, -a) \text{ \& } x \in (-\infty, 0) \\ |x-1|+1 & x \in \phi \\ (x+a-1)^2+b & x \in [-a, \infty) \text{ \& } x \in (-\infty, 0) \\ (|x-1|-1)^2+b & x \in \mathbb{R} \text{ \& } x \in [0, \infty) \end{cases}$$

$$g[f(x)] = \begin{cases} x+a+1 & x \in (-\infty, -a) \\ (x+a-1)^2+b & x \in [-a, 0) \\ (|x-1|-1)^2+b & x \in [0, \infty) \end{cases}$$

$g(f(x))$  is continuous

$$\begin{aligned} \text{at } x = -a & \quad \& \quad \text{at } x = 0 \\ 1 = b + 1 & \quad \& \quad (a - 1)^2 + b = b \\ b = 0 & \quad \& \quad a = 1 \\ \Rightarrow a + b = 1 \end{aligned}$$

6. Let  $\frac{1}{16}$ ,  $a$  and  $b$  be in G.P. and  $\frac{1}{a}, \frac{1}{b}, 6$  be in A.P., where  $a, b > 0$ . Then  $72(a + b)$  is equal to \_\_\_\_\_.

**Official Ans by NTA (14)**

**Sol.**  $a^2 = \frac{b}{16} \Rightarrow \frac{1}{b} = \frac{1}{16a^2}$

$$\frac{2}{b} = \frac{1}{a} + 6$$

$$\frac{1}{8a^2} = \frac{1}{a} + 6$$

$$\frac{1}{a^2} - \frac{8}{a} - 48 = 0$$

$$\frac{1}{a} = 12, -4 \Rightarrow a = \frac{1}{12}, -\frac{1}{4}$$

$$a = \frac{1}{12}, a > 0$$

$$b = 16a^2 = \frac{1}{9}$$

$$\Rightarrow 72(a + b) = 6 + 8 = 14$$

7. In  $\Delta ABC$ , the lengths of sides AC and AB are 12 cm and 5 cm, respectively. If the area of  $\Delta ABC$  is  $30 \text{ cm}^2$  and  $R$  and  $r$  are respectively the radii of circumcircle and incircle of  $\Delta ABC$ , then the value of  $2R + r$  (in cm) is equal to \_\_\_\_\_.

**Official Ans by NTA (15)**

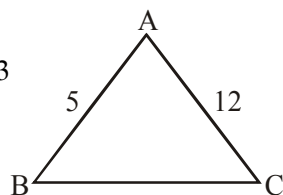
**Sol.**  $\Delta = \frac{1}{2} \cdot 5 \cdot 12 \cdot \sin A = 30$

$$\sin A = 1$$

$$A = 90^\circ \Rightarrow BC = 13$$

$$BC = 2R = 13$$

$$r = \frac{\Delta}{s} = \frac{30}{12} = 2.5$$



8. Let  $n$  be a positive integer. Let

$$A = \sum_{k=0}^n (-1)^k n C_k \left[ \left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k + \left(\frac{7}{8}\right)^k + \left(\frac{15}{16}\right)^k + \left(\frac{31}{32}\right)^k \right]$$

If  $63A = 1 - \frac{1}{2^{30}}$ , then  $n$  is equal to \_\_\_\_\_.

**Official Ans by NTA (6)**

**Sol.**  $A = \sum_{k=0}^n n C_k \left[ \left(-\frac{1}{2}\right)^k + \left(\frac{-3}{4}\right)^k + \left(\frac{-7}{8}\right)^k + \left(\frac{-15}{16}\right)^k + \left(\frac{-31}{32}\right)^k \right]$

$$A = \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n + \left(1 - \frac{15}{16}\right)^n + \left(1 - \frac{31}{32}\right)^n$$

$$A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} + \frac{1}{32^n}$$

$$A = \frac{1}{2^n} \left( \frac{1 - \left(\frac{1}{2}\right)^{5n}}{1 - \frac{1}{2^n}} \right) \Rightarrow A = \frac{\left(1 - \frac{1}{2^{5n}}\right)}{(2^n - 1)}$$

$$(2^n - 1)A = 1 - \frac{1}{2^{5n}}, \text{ Given } 63A = 1 - \frac{1}{2^{30}}$$

Clearly  $5n = 30$

$$n = 6$$

9. Let  $\vec{c}$  be a vector perpendicular to the vectors  $\vec{a} = \hat{i} + \hat{j} - \hat{k}$  and  $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ .

If  $\vec{c} \cdot (\hat{i} + \hat{j} + 3\hat{k}) = 8$  then the value of  $\vec{c} \cdot (\vec{a} \times \vec{b})$  is equal to \_\_\_\_\_.

**Official Ans by NTA (28)**

**Sol.**  $\vec{c} = \lambda(\vec{a} \times \vec{b})$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$(\vec{a} \times \vec{b}) = 3\hat{i} - 2\hat{j} + \hat{k}$$

.....

$$\Rightarrow \lambda(4) = 8 \Rightarrow \lambda = 2$$

$$\vec{c} = 2(\vec{a} \times \vec{b})$$

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = 2 |\vec{a} \times \vec{b}|^2 = 28$$

10. Let

$$S_n(x) = \log_{a^{1/2}} x + \log_{a^{1/3}} x + \log_{a^{1/6}} x$$

$$+ \log_{a^{1/11}} x + \log_{a^{1/18}} x + \log_{a^{1/27}} x + \dots$$

up to n-terms, where  $a > 1$ . If  $S_{24}(x) = 1093$  and  $S_{12}(2x) = 265$ , then value of a is equal to \_\_\_\_\_.

**Official Ans by NTA (16)**

Sol.  $S_n(x) = (2+3+6+11+18+27+\dots+n\text{-terms})\log_a x$

Let  $S_1 = 2 + 3 + 6 + 11 + 18 + 27 + \dots + T_n$

$$S_1 = 2 + 3 + 6 + \dots + T_n$$

---


$$T_n = 2 + 1 + 3 + 5 + \dots + n \text{ terms}$$

$$T_n = 2 + (n - 1)^2$$

$$S_1 = \Sigma T_n = 2n + \frac{(n-1)n(2n-1)}{6}$$

$$\Rightarrow S_n(x) = \left( 2n + \frac{n(n-1)(2n-1)}{6} \right) \log_a x$$

$$S_{24}(x) = 1093 \text{ (Given)}$$

$$\log_a x \left( 48 + \frac{23 \cdot 24 \cdot 47}{6} \right) = 1093$$

$$\log_a x = \frac{1}{4} \dots (1)$$

$$S_{12}(2x) = 265$$

$$S_{12}(2x) = 265$$

$$\log_a(2x) \left( 24 + \frac{11 \cdot 12 \cdot 23}{6} \right) = 265$$

$$\log_a 2x = \frac{1}{2} \dots (2)$$

$$(2) - (1)$$

$$\log_a 2x - \log_a x = \frac{1}{4}$$

$$\log_a 2 = \frac{1}{4} \Rightarrow a = 16$$