FINAL JEE–MAIN EXAMINATION – MARCH, 2021

(Held On Wednesday 17th March, 2021) TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let f : R → R be defined as $f(x) = e^{-x}\sin x$. If F : [0, 1] → R is a differentiable function such that $F(x)=\int_0^x f(t)\,dt$, then the value of $\int_0^1 (F'(x)+f(x))e^x dx$ lies in the interval

(1) $\left[\frac{327}{360},\frac{329}{360}\right]$ (2) $\left[\frac{330}{360},\frac{331}{360}\right]$

(3) $\left[\frac{331}{360},\frac{334}{360}\right]$ (4) $\left[\frac{335}{360},\frac{336}{360}\right]$

Official Ans. by NTA (2)

Sol. $f(x)=e^{-x}\sin x$

Now, $F(x)=\int_0^x f(t)dt \Rightarrow F'(x) = f(x)$

$$I=\int_0^1 (F'(x)+f(x))e^x\,dx = \int_0^1 (f(x)+f(x))\cdot e^x\,dx$$

$$= 2\int_0^1 f(x)\cdot e^x\,dx = 2\int_0^1 e^{-x}\sin x\cdot e^x\,dx$$

$$= 2\int_0^1 \sin x\,dx$$

$= 2(1 - \cos 1)$

$$I=2\left\{1-\left(1-\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\frac{1}{\underline{|8}}.........\right)\right\}$$

$$I=1-\frac{2}{\underline{|4}}+\frac{2}{\underline{|6}}-\frac{2}{\underline{|9}}+......$$

$$1-\frac{2}{\underline{|4}}<I<1-\frac{2}{\underline{|4}}+\frac{2}{\underline{|6}}$$

$$\frac{11}{12}<I<\frac{331}{360}$$

$$\Rightarrow I\in\left[\frac{11}{12},\frac{331}{360}\right]$$

$$\left[\frac{330}{360},\frac{331}{360}\right]$$

2. If the integral $\int_0^{10}\frac{[\sin 2\pi x]}{e^{x-[x]}}dx=\alpha e^{-1}+\beta e^{-\frac{1}{2}}+\gamma$, where α, β, γ are integers and [x] denotes the greatest integer less than or equal to x, then the value of α + β + γ is equal to :

(1) 0 (2) 20 (3) 25 (4) 10

Official Ans. by NTA (1)

Sol. Let $I=\int_0^{10}\frac{[\sin 2\pi x]}{e^{x-[x]}}dx = \int_0^{10}\frac{[\sin 2\pi x]}{e^{\{x\}}}dx$

Function $f(x)=\frac{[\sin 2\pi x]}{e^{\{x\}}}$ is periodic with period '1'

Therefore

$$I=10\int_0^1\frac{[\sin 2\pi x]}{e^{\{x\}}}dx$$

$$=10\int_0^1\frac{[\sin 2\pi x]}{e^x}dx$$

$$=10\left(\int_0^{1/2}\frac{[\sin 2\pi x]}{e^x}dx+\int_{1/2}^1\frac{[\sin 2\pi x]}{e^x}dx\right)$$

$$=10\left(0+\int_{1/2}^1\frac{(-1)}{e^x}dx\right)$$

$$=-10\int_{1/2}^1 e^{-x}\,dx$$

$$=10\left(e^{-1}-e^{-1/2}\right)$$

Now,

$10\cdot e^{-1}-10\cdot e^{-1/2}=\alpha e^{-1}+\beta e^{-1/2}+\gamma$ (given)

$\Rightarrow \alpha = 10,\ \beta = -10,\ \gamma = 0$

$\Rightarrow \alpha + \beta + \gamma = 0$ **Ans. (1)**

3. Let y = y(x) be the solution of the differential equation cosx (3sinx + cosx + 3)dy = (1 + y sinx (3sinx + cosx + 3))dx, $0 \le x \le \frac{\pi}{2}$, y(0) = 0. Then, $y\left(\frac{\pi}{3}\right)$ is equal to:

(1) $2\log_e\left(\frac{2\sqrt{3}+9}{6}\right)$ (2) $2\log_e\left(\frac{2\sqrt{3}+10}{11}\right)$

(3) $2\log_e\left(\frac{\sqrt{3}+7}{2}\right)$ (4) $2\log_e\left(\frac{3\sqrt{3}-8}{4}\right)$

Official Ans. by NTA (2)

Sol. $\cos x(3\sin x+\cos x+3)dy$

$= (1+y\sin x(3\sin x+\cos x+3))dx$

$$\frac{dy}{dx}-(\tan x)y=\frac{1}{(3\sin x+\cos x+3)\cos x}$$

$$\text{I.F.}=e^{\int -\tan x\,dx}=e^{\ell n|\cos x|}=|\cos x|$$

$$=\cos x\ \forall\ x\in\left[0,\frac{\pi}{2}\right)$$

Solution of D.E.

$$y(\cos x)=\int(\cos x)\cdot\frac{1}{\cos x(3\sin x+\cos x+3)}dx+C$$

$$y(\cos x)=\int\frac{dx}{3\sin x+\cos x+3}dx+C$$

$$y(\cos x)=\int\frac{\left(\sec^2\frac{x}{2}\right)}{2\tan^2\frac{x}{2}+6\tan\frac{x}{2}+4}dx+C$$

Now

$$\text{Let } I_1=\int\frac{\left(\sec^2\frac{x}{2}\right)}{2\left(\tan^2\frac{x}{2}+3\tan\frac{x}{2}+2\right)}dx+C$$

Put $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx=dt$

$$I_1=\int\frac{dt}{t^3+3t+2}=\int\frac{dt}{(t+2)(t+1)}$$

$$=\int\left(\frac{1}{t+1}-\frac{1}{t+2}\right)dt$$

$$=\ell n\left|\left(\frac{t+1}{t+2}\right)\right| = \ell n\left|\left(\frac{\tan\frac{x}{2}+1}{\frac{x}{}}\right)\right|$$

So solution of D.E.

$$y(\cos x)=\ell n\left|\frac{1+\tan\frac{x}{2}}{2+\tan\frac{x}{2}}\right|+C$$

$$\Rightarrow y(\cos x)=\ell n\left(\frac{1+\tan\frac{x}{2}}{2+\tan\frac{x}{2}}\right)+C \quad \text{for } 0\le x<\frac{\pi}{2}$$

Now, it is given y(0) = 0

$$\Rightarrow 0=\ell n\left(\frac{1}{2}\right)+C \qquad \Rightarrow \boxed{C=\ell n2}$$

$$\Rightarrow y(\cos x)=\ell n\left(\frac{1+\tan\frac{x}{2}}{2+\tan\frac{x}{2}}\right)+\ell n2$$

For $x=\frac{\pi}{3}$

$$y\left(\frac{1}{2}\right)=\ell n\left(\frac{1+\frac{1}{\sqrt{3}}}{2+\frac{1}{\sqrt{3}}}\right)+\ell n2$$

$$y=2\ell n\left(\frac{2\sqrt{3}+10}{11}\right)$$ **Ans.(2)**

4. The value of $\sum_{r=0}^{6}\left({}^6C_r\cdot{}^6C_{6-r}\right)$ is equal to :

(1) 1124 (2) 1324 (3) 1024 (4) 924

Official Ans. by NTA (4)

Sol. $\sum_{r=0}^{6}{}^6C_r\cdot{}^6C_{6-r}$

$= {}^6C_0\cdot{}^6C_6+{}^6C_1\cdot{}^6C_5+\ldots\ldots+{}^6C_6\cdot{}^6C_0$

Now,

$(1+x)^6(1+x)^6$

$=\left({}^6C_0+{}^6C_1x+{}^6C_2x^2+\ldots\ldots+{}^6C_6x^6\right)$

$\left({}^6C_0+{}^6C_1x+{}^6C_2x^2+\ldots\ldots+{}^6C_6x^6\right)$

Comparing coefficeint of x^6 both sides

${}^6C_0\cdot{}^6C_6+{}^6C_1+{}^6C_5+\ldots\ldots+{}^6C_6\cdot{}^6C_0={}^{12}C_6$

5. The value of $\lim_{n\to\infty}\frac{[r]+[2r]+.....+[nr]}{n^2}$, where r is non-zero real number and [r] denotes the greatest integer less than or equal to r, is equal to :

(1) $\frac{r}{2}$ (2) r (3) 2r (4) 0

Official Ans. by NTA (1)

Sol. We know that

$$r \le [r] < r + 1$$

and $2r \le [2r] < 2r + 1$

$$3r \le [3r] < 3r + 1$$

$$\vdots \quad \vdots \quad \vdots$$

$$nr \le [nr] < nr + 1$$

$r + 2r + ...+ nr$

$\le [r] + [2r] +....+ [nr] < (r + 2r +....+ nr) + n$

$$\frac{\frac{n(n+1)}{2}\cdot r}{n^2} \le \frac{[r]+[2r]+......+[nr]}{n^2} < \frac{\frac{n(n+1)}{2}r+n}{n^2}$$

Now,

$$\lim_{n\to\infty}\frac{n(n+1)\cdot r}{2\cdot n^2}=\frac{r}{2}$$

and $\lim_{n\to\infty}\frac{\frac{n(n+1)r}{2}+n}{n^2}=\frac{r}{2}$

So, by Sandwich Theorem, we can conclude that

$$\lim_{n\to\infty}\frac{[r]+[2r]+......+[nr]}{n^2}=\frac{r}{2}$$

Ans. (1)

6. The number of solutions of the equation $\sin^{-1}\left[x^2+\frac{1}{3}\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]=x^2$, for $x \in [-1, 1]$, and [x] denotes the greatest integer less than or equal to x, is :

(1) 2 (2) 0

(3) 4 (4) Infinite

Official Ans. by NTA (2)

Sol. Given equation

$$\sin^{-1}\left[x^2+\frac{1}{3}\right]+\cos^{-1}\left[x^2-\frac{2}{3}\right]=x^2$$

$-1 \le x^2+\frac{1}{3}<2 \Rightarrow \frac{-4}{3}\le x^2<\frac{5}{3}$

$\Rightarrow \boxed{0\le x^2<\frac{5}{3}}$ ....(1)

and $\cos^{-1}\left[x^2-\frac{2}{3}\right]$ is defined if

$-1 \le x^2-\frac{2}{3}<2 \Rightarrow \frac{-1}{3}\le x^2<\frac{8}{3}$

$\Rightarrow \boxed{0\le x^2<\frac{8}{3}}$ ....(2)

So, form (1) and (2) we can conclude

$$\boxed{0\le x^2<\frac{5}{3}}$$

Case - I if $0\le x^2<\frac{2}{3}$

$\sin^{-1}(0)+\cos^{-1}(-1)=x^2$

$\Rightarrow x + \pi = x^2$

$\Rightarrow x^2 = \pi$

but $\pi \notin \left[0,\frac{2}{3}\right)$

$\Rightarrow$ No value of 'x'

Case - II if $\frac{2}{3}\le x^2<\frac{5}{3}$

$\sin^{-1}(1)+\cos^{-1}(0)=x^2$

$\Rightarrow \frac{\pi}{2}+\frac{\pi}{2}=x^2$

$\Rightarrow x^2 = \pi$

but $\pi \notin \left[\frac{2}{3},\frac{5}{3}\right)$

$\Rightarrow$ No value of 'x'

So, number of solutions of the equation is zero.

Ans.(2)

7. Let a computer program generate only the digits 0 and 1 to form a string of binary numbers with probability of occurrence of 0 at even places be $\frac{1}{2}$ and probability of occurrence of 0 at the odd place be $\frac{1}{3}$. Then the probability that '10' is followed by '01' is equal to :

(1) $\frac{1}{18}$ (2) $\frac{1}{3}$ (3) $\frac{1}{6}$ (4) $\frac{1}{9}$

Sol. $\underset{\text{odd place}}{1}\ \underset{\text{even place}}{0}\ \underset{\text{odd place}}{0}\ \underset{\text{even place}}{1}$

or $\underset{\text{even place}}{1}\ \underset{\text{odd place}}{0}\ \underset{\text{even place}}{0}\ \underset{\text{odd place}}{1}$

$\Rightarrow \left(\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3}\right)+\left(\frac{2}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)$

$\Rightarrow \frac{1}{9}$

8. The number of solutions of the equation $x + 2\tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ is :

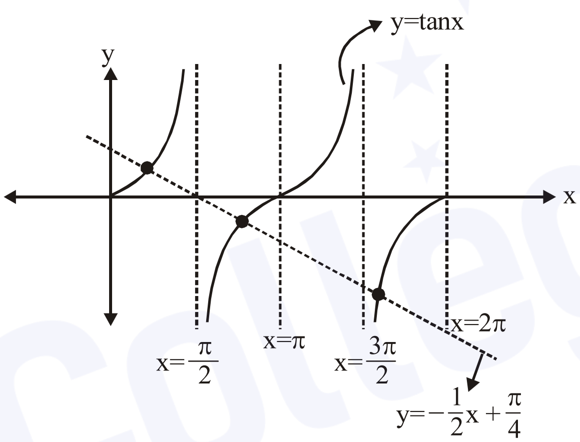
(1) 3 (2) 4 (3) 2 (4) 5

Official Ans. by NTA (1)

Sol. $x + 2\tan x = \frac{\pi}{2}$

$\Rightarrow 2\tan x = \frac{\pi}{2} - x$

$\Rightarrow \tan x = -\frac{1}{2}x + \frac{\pi}{4}$



Number of soluitons of the given eauation is '3'.

Ans. (1)

9. Let S_1, S_2 and S_3 be three sets defined as

$S_1 = \{z \in \mathbb{C} : |z-1| \le \sqrt{2}\}$

$S_2 = \{z \in \mathbb{C} : \mathrm{Re}((1-i)z) \ge 1\}$

$S_3 = \{z \in \mathbb{C} : \mathrm{Im}(z) \le 1\}$

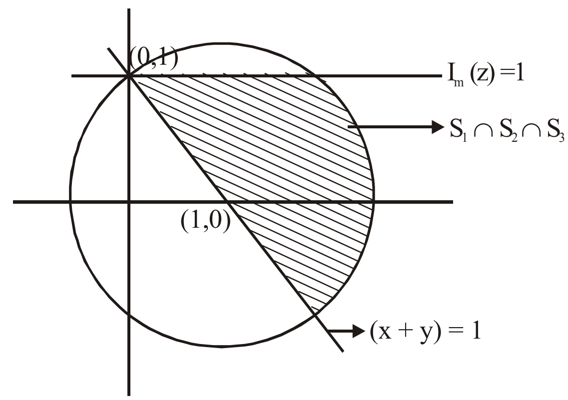
Then the set $S_1 \cap S_2 \cap S_3$

(1) is a singleton

(2) has exactly two elements

(3) has infinitely many elements

Sol. For $|z-1| \le \sqrt{2}$, z lies on and inside the circle of radius $\sqrt{2}$ units and centre (1, 0).



For S_2

Let z = x + iy

Now, (1 – i) (z) = (1 – i) (x + iy)

Re((1 – i)z) = x + y

$\Rightarrow x + y \ge 1$

$\Rightarrow S_1 \cap S_2 \cap S_3$ has infinity many elements

Ans. (3)

10. If the curve y = y(x) is the solution of the differential equation

$2(x^2 + x^{5/4})dy - y(x + x^{1/4})dx = 2x^{9/4}\,dx$, $x > 0$

which passes through the point $\left(1, 1-\frac{4}{3}\log_e 2\right)$, then the value of y(16) is equal to :

(1) $4\left(\frac{31}{3}+\frac{8}{3}\log_e 3\right)$ (2) $\left(\frac{31}{3}+\frac{8}{3}\log_e 3\right)$

(3) $4\left(\frac{31}{3}-\frac{8}{3}\log_e 3\right)$ (4) $\left(\frac{31}{3}-\frac{8}{3}\log_e 3\right)$

Official Ans. by NTA (3)

Sol. $\frac{dy}{dx} - \frac{y}{2x} = \frac{x^{9/4}}{x^{5/4}(x^{3/4}+1)}$

$IF = e^{-\int\frac{dx}{2d}} = e^{-\frac{1}{2}\ln x} = \frac{1}{x^{1/2}}$

$y.x^{-1/2} = \int \frac{x^{9/4}\cdot x^{-1/2}}{x^{5/4}\left(x^{3/4}+1\right)}dx$

$\int \frac{x^{1/2}}{3/4}dx$

$x = t^4 \Rightarrow dx = 4t^3\,dt$

$$\int \frac{t^2 \cdot 4t^3\,dt}{(t^3+1)}$$

$$4\int \frac{t^2(t^3+1-1)}{(t^3+1)}dt$$

$$4\int t^2 dt - 4\int \frac{t^2}{t^3+1}dt$$

$$\frac{4t^3}{3} - \frac{4}{3}\ln(t^3+1) + C$$

$$yx^{-1/2} = \frac{4x^{3/4}}{3} - \frac{4}{3}\ln(x^{3/4}+1) + C$$

$$1 - \frac{4}{3}\log_e 2 = \frac{4}{3} - \frac{4}{3}\log_e 2 + C$$

$$\Rightarrow C = -\frac{1}{3}$$

$$y = \frac{4}{3}x^{5/4} - \frac{4}{3}\sqrt{x}\ln(x^{3/4}+1) - \frac{\sqrt{x}}{3}$$

$$y(16) = \frac{4}{3}\times 32 - \frac{4}{3}\times 4\ln 9 - \frac{4}{3}$$

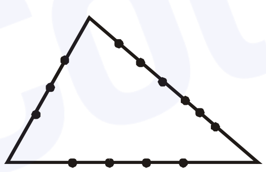
$$= \frac{124}{3} - \frac{32}{3}\ln 3 = 4\left(\frac{31}{3} - \frac{8}{3}\ln 3\right)$$

11. If the sides AB, BC and CA of a triangle ABC have 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to :

(1) 364 (2) 240 (3) 333 (4) 360

Official Ans. by NTA (3)

Sol.



Total Number of triangles formed

$= {}^{14}C_3 - {}^{3}C_3 - {}^{5}C_3 - {}^{6}C_3$

$= 333$ **Option (3)**

12. If x, y, z are in arithmetic progression with common difference d, $x \neq 3d$, and the determinant of the matrix $\begin{bmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{bmatrix}$ is zero, then the value of k^2 is

Sol. $\begin{vmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0$

$R_2 \to R_1 + R_3 - 2R_2$

$$\Rightarrow \begin{vmatrix} 3 & 4\sqrt{2} & x \\ 0 & k-6\sqrt{2} & 0 \\ 5 & k & z \end{vmatrix} = 0$$

$\Rightarrow (k - 6\sqrt{2})(3z - 5x) = 0$

if $3z - 5x = 0 \Rightarrow 3(x + 2d) - 5x = 0$

$\Rightarrow x = 3d$ (Not possible)

$\Rightarrow k = 6\sqrt{2} \quad \Rightarrow k^2 = 72$ **Option (1)**

13. Let O be the origin. Let $\overrightarrow{OP} = x\hat{i} + y\hat{j} - \hat{k}$ and $\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$, $x, y \in R$, $x > 0$, be such that $|\overrightarrow{PQ}| = \sqrt{20}$ and the vector $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$. If $\overrightarrow{OR} = 3\hat{i} + z\hat{j} - 7\hat{k}$, $z \in R$, is coplanar with $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the value of $x^2 + y^2 + z^2$ is equal to

(1) 7 (2) 9 (3) 2 (4) 1

Official Ans. by NTA (2)

Sol. $\overrightarrow{OP} \perp \overrightarrow{OQ}$

$\Rightarrow -x + 2y - 3x = 0$

$\Rightarrow y = 2x$ .....(i)

$|\overrightarrow{PQ}|^2 = 20$

$\Rightarrow (x + 1)^2 + (y - 2)^2 + (1 + 3x)^2 = 20$

$\Rightarrow x = 1$

$\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ are coplanar.

$$\Rightarrow \begin{vmatrix} x & y & -1 \\ -1 & 2 & 3x \\ 3 & z & -7 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 3 & z & -7 \end{vmatrix} = 0$$

$\Rightarrow 1(-14 - 3z) - 2(7 - 9) - 1(-z - 6) = 0$

$\Rightarrow z = -2$

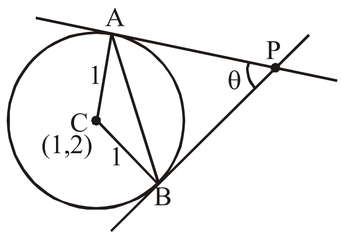
$\therefore x^2 + y^2 + z^2 = 1 + 4 + 4 = 9$ **Option (2)**

14. Two tangents are drawn from a point P to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$, such that the angle between these tangents is $\tan^{-1}\left(\frac{12}{5}\right)$, where $\tan^{-1}\left(\frac{12}{5}\right) \in (0, \pi)$. If the centre of the circle is denoted by C and these tangents touch the circle at points A and B, then the ratio of the areas of ΔPAB and ΔCAB is :

(1) 11 : 4 (2) 9 : 4 (3) 3 :1 (4) 2 : 1

Official Ans. by NTA (2)

Sol.



$$\tan\theta = \frac{12}{5}$$

$$PA = \cot\frac{\theta}{2}$$

$$\therefore \text{ area of } \Delta PAB = \frac{1}{2}(PA)^2 \sin\theta = \frac{1}{2}\cot^2\frac{\theta}{2}\sin\theta$$

$$= \frac{1}{2}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\sin\theta$$

$$= \frac{1}{2}\left(\frac{1+\frac{5}{13}}{1-\frac{5}{13}}\right)\left(\frac{12}{13}\right) = \frac{1}{2}\frac{18}{18}\times\frac{2}{13} = \frac{27}{26}$$

$$\text{area of } \Delta CAB = \frac{1}{2}\sin\theta = \frac{1}{2}\left(\frac{12}{13}\right) = \frac{6}{13}$$

$$\therefore \frac{\text{area of } \Delta PAB}{\text{area of } \Delta CAB} = \frac{9}{4}$$ **Option (2)**

15. Consider the function $f : R \to R$ defined by

$$f(x) = \begin{cases} \left(2-\sin\left(\frac{1}{x}\right)\right)|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$$. Then f is :

(1) monotonic on $(-\infty, 0) \cup (0, \infty)$
(2) not monotonic on $(-\infty, 0)$ and $(0, \infty)$
(3) monotonic on $(0, \infty)$ only
(4) monotonic on $(-\infty, 0)$ only

Official Ans. by NTA (2)

Sol. $$f(x) = \begin{cases} -x\left(2-\sin\left(\frac{1}{x}\right)\right) & x<0 \\ 0 & x=0 \\ x\left(2-\sin\left(\frac{1}{x}\right)\right) & \end{cases}$$

$$f'(x) = \begin{cases} -\left(2-\sin\frac{1}{x}\right) - x\left(-\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)\right) & x<0 \\ \left(2-\sin\frac{1}{x}\right) + x\left(-\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)\right) & x>0 \end{cases}$$

$$f'(x) = \begin{cases} -2+\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} & x<0 \\ 2-\sin\frac{1}{x} + \frac{1}{x}\cos\frac{1}{x} & x>0 \end{cases}$$

$f'(x)$ is an oscillating function which is non-monotonic in $(-\infty, 0) \cup (0, \infty)$.

Option (2)

16. Let L be a tangent line to the parabola $y^2 = 4x - 20$ at (6, 2). If L is also a tangent to the ellipse $\frac{x^2}{2} + \frac{y^2}{b} = 1$, then the value of b is equal to :

(1) 11 (2) 14 (3) 16 (4) 20

Official Ans. by NTA (2)

Sol. Tangent to parabola
$2y = 2(x + 6) - 20$
$\Rightarrow y = x - 4$
Condition of tangency for ellipse.
$16 = 2(1)^2 + b$
$\Rightarrow b = 14$ **Option (2)**

17. The value of the limit $\lim_{\theta\to 0}\frac{\tan\left(\pi\cos^2\theta\right)}{\sin\left(2\pi\sin^2\theta\right)}$ is equal to :

(1) $-\frac{1}{2}$ (2) $-\frac{1}{4}$ (3) 0 (4) $\frac{1}{4}$

Official Ans. by NTA (1)

Sol. $$\lim_{\theta\to 0}\frac{\tan\left(\pi\left(1-\sin^2\theta\right)\right)}{\sin\left(2\pi\sin^2\theta\right)}$$

$$= \lim_{\theta\to 0}\frac{-\tan\left(\pi\sin^2\theta\right)}{\sin\left(2\pi\sin^2\theta\right)}$$

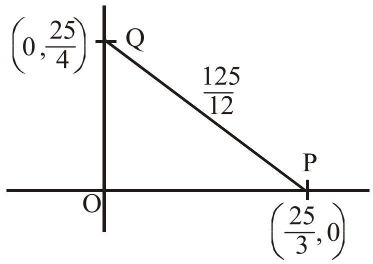
$$= \lim_{\theta\to 0} -\left(\frac{\tan\left(\pi\sin^2\theta\right)}{\pi\sin^2\theta}\right)\left(\frac{2\pi\sin^2\theta}{\sin\left(2\pi\sin^2\theta\right)}\right)\times\frac{1}{2}$$

18. Let the tangent to the circle $x^2 + y^2 = 25$ at the point R(3, 4) meet x-axis and y-axis at point P and Q, respectively. If r is the radius of the circle passing through the origin O and having centre at the incentre of the triangle OPQ, then r^2 is equal to

(1) $\frac{529}{64}$ (2) $\frac{125}{72}$ (3) $\frac{625}{72}$ (4) $\frac{585}{66}$

Official Ans. by NTA (3)

Sol. Tangent to circle $3x + 4y = 25$



OP + OQ + OR = 25

$$\text{Incentre} = \left(\frac{\frac{25}{4}\times\frac{25}{3}}{25},\frac{\frac{25}{4}\times\frac{25}{3}}{25}\right)$$

$$= \left(\frac{25}{12},\frac{25}{12}\right)$$

$$\therefore\ r^2 = 2\left(\frac{25}{12}\right)^2 = 2\times\frac{625}{144} = \frac{625}{72}$$

Option (3)

19. If the Boolean expression $(p\wedge q)\circledast(p\otimes q)$ is a tautology, then $\circledast$ and $\otimes$ are respectively given by

(1) $\rightarrow, \rightarrow$ (2) $\wedge, \vee$ (3) $\vee, \rightarrow$ (4) $\wedge, \rightarrow$

Official Ans. by NTA (1)

Sol. **Option (1)**

$(p\wedge q)\longrightarrow(p\rightarrow q)$

$= \sim(p\wedge q)\vee(\sim p\vee q)$

$= (\sim p\vee \sim q)\vee(\sim p\vee q)$

$= \sim p\vee(\sim q\vee q)$

$= \sim p\vee t$

$= t$

Option (2)

$(p\wedge q)\wedge(p\vee q) = (p\wedge q)$ (Not a tautology)

Option (3)

$(p\wedge q)\vee(p\rightarrow q)$

Option (4)

$= (p\wedge q)\wedge(\sim p\vee q)$

$= p\wedge q$ (Not a tautology)

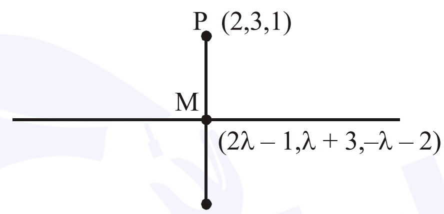
Option (1)

20. If the equation of plane passing through the mirror image of a point (2, 3, 1) with respect to line $\frac{x+1}{2}=\frac{y-3}{1}=\frac{z+2}{-1}$ and containing the line $\frac{x-2}{3}=\frac{1-y}{2}=\frac{z+1}{1}$ is $\alpha x + \beta y + \gamma z = 24$, then $\alpha + \beta + \gamma$ is equal to :

(1) 20 (2) 19 (3) 18 (4) 21

Official Ans. by NTA (2)

Sol. Line $\frac{x+1}{2}=\frac{y-3}{1}=\frac{z+2}{-1}$



$\overrightarrow{PM} = (2\lambda-3, \lambda, -\lambda-3)$

$\overrightarrow{PM}\perp\left(2\hat{i}+\hat{j}-\hat{k}\right)$

$4\lambda - 6 + \lambda + \lambda + 3 = 0 \Rightarrow \lambda = \frac{1}{2}$

$\therefore\ M \equiv \left(0,\frac{7}{2},\frac{-5}{2}\right)$

$\therefore$ Reflection (–2, 4, –6)

$$\text{Plane : }\begin{vmatrix} x-2 & y-1 & z+1 \\ 3 & -2 & 1 \\ 4 & -3 & 5 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(-10+3) - (y-1)(15-4) + (z+1)(-1) = 0$

$\Rightarrow -7x + 14 - 11y + 11 - z - 1 = 0$

$\Rightarrow 7x + 11y + z = 24$

$\therefore\ \alpha = 7, \beta = 11, \gamma = 1$

$\alpha + \beta + \gamma = 19$ **Option (2)**

SECTION-B

1. If 1, $\log_{10}(4^x - 2)$ and $\log_{10}\left(4^x + \frac{18}{5}\right)$ are in arithmetic progression for a real number x, then the value of the determinant

$$\begin{vmatrix} 2\left(x-\frac{1}{2}\right) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$$

is equal to :

Sol. $2\log_{10}(4^x-2)=1+\log_{10}\left(4^x+\frac{18}{5}\right)$

$(4^x-2)^2=10\left(4^x+\frac{18}{5}\right)$

$(4^x)^2+4-4(4^x)-32=0$

$(4^x-16)(4^x+2)=0$

$4^x=16$

$x=2$

$$\begin{vmatrix}3&1&4\\1&0&2\\2&1&0\end{vmatrix}=3(-2)-1(0-4)+4(1)$$

$=-6+4+4=2$

2. Let $f:[-1,1]\to R$ be defined as $f(x)=ax^2+bx+c$ for all $x\in[-1,1]$, where $a, b, c\in R$ such that $f(-1)=2$, $f'(-1)=1$ and for $x\in(-1,1)$ the maximum value of $f''(x)$ is $\frac{1}{2}$. If $f(x)\le\alpha$, $x\in[-1,1]$, then the least value of α is equal to ______.

Official Ans. by NTA (5)

Sol. $f:[-1,1]\to R$

$f(x)=ax^2+bx+c$

$f(-1)=a-b+c=2$ ...(1)

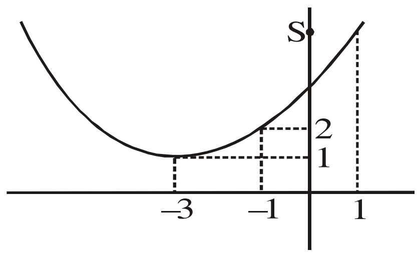
$f'(-1)=-2a+b=1$ ...(2)

$f''(x)=2a$

$\Rightarrow$ Max. value of $f''(x)=2a=\frac{1}{2}$

$\Rightarrow a=\frac{1}{4};\quad b=\frac{3}{2};\quad c=\frac{13}{4}$

$\therefore\quad f(x)=\frac{x^2}{4}+\frac{3}{2}x+\frac{13}{4}$



3. Let $f:[-3,1]\to R$ be given as

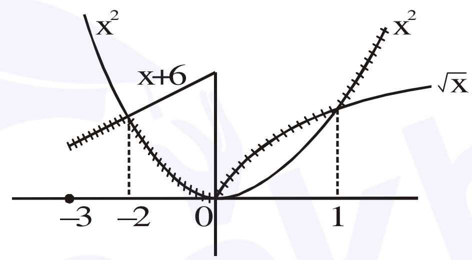
$$f(x)=\begin{cases}\min\{(x+6),x^2\}, & -3\le x\le 0\\ \max\{\sqrt{x},x^2\}, & 0\le x\le 1.\end{cases}$$

If the area bounded by $y=f(x)$ and x-axis is A, then the value of 6A is equal to ______.

Official Ans. by NTA (41)

Sol. $f:[-3,1]\to R$

$$f(x)=\begin{cases}\min\{(x+6),x^2\}, & -3\le x\le 0\\ \max\{\sqrt{x},x^2\}, & 0\le x\le 1\end{cases}$$



area bounded by $y=f(x)$ and x-axis

$$=\int_{-3}^{-2}(x+6)dx+\int_{-2}^{0}x^2dx+\int_{0}^{1}\sqrt{x}\,dx$$

$A=\frac{41}{6}$

$6A=41$

4. Let $\tan\alpha$, $\tan\beta$ and $\tan\gamma$; $\alpha,\beta,\gamma\ne\frac{(2n-1)\pi}{2}$, $n\in N$ be the slopes of three line segments OA, OB and OC, respectively, where O is origin.If circumcentre of ΔABC coincides with origin and its orthocentre lies on y-axis, then the value of $\left(\frac{\cos3\alpha+\cos3\beta+\cos3\gamma}{\cos\alpha\cos\beta\cos\gamma}\right)^2$ is equal to :

Official Ans. by NTA (144)

Sol. Since orthocentre and circumcentre both lies on y-axis

$\Rightarrow$ Centroid also lies on y-axis

$\Rightarrow \Sigma\cos\alpha = 0$

$\cos\alpha + \cos\beta + \cos\gamma = 0$

$\Rightarrow \cos^3\alpha + \cos^3\beta + \cos^3\gamma = 3\cos\alpha\cos\beta\cos\gamma$

$\therefore \dfrac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}$

$= \dfrac{4(\cos^3\alpha + \cos^3\beta + \cos^3\gamma) - 3(\cos\alpha + \cos\beta + \cos\gamma)}{\cos\alpha\cos\beta\cos\gamma}$

$= 12$

5. Consider a set of 3n numbers having variance 4. In this set, the mean of first 2n numbers is 6 and the mean of the remaining n numbers is 3. A new set is constructed by adding 1 into each of first 2n numbers, and subtracting 1 from each of the remaining n numbers. If the variance of the new set is k, then 9k is equal to ______.

Official Ans. by NTA (68)

Sol. Let number be $a_1, a_2, a_3, \ldots\ldots a_{2n}, b_1, b_2, b_3 \ldots\ldots b_n$

$$\sigma^2 = \frac{\sum a^2 + \sum b^2}{3n} - (5)^2$$

$$\Rightarrow \sum a^2 + \sum b^2 = 87n$$

Now, distribution becomes

$a_1 + 1, a_2 + 1, a_3 + 1, \ldots\ldots a_{2n} + 1, b_1 - 1, b_2 - 1 \ldots\ldots b_n - 1$

Variance

$$= \frac{\sum(a+1)^2 + \sum(b-1)^2}{3n} - \left(\frac{12n + 2n + 3n - n}{3n}\right)^2$$

$$= \frac{\left(\sum a^2 + 2n + 2\sum a\right) + \left(\sum b^2 + n - 2\sum b\right)}{3n}$$

$$= \frac{\left(\sum a^2 + 2n + 2\sum a\right) + \left(\sum b^2 + n - 2\sum b\right)}{3n} - \left(\frac{16}{3}\right)^2$$

$$= \frac{87n + 3n + 2(12n) - 2(3n)}{3n} - \left(\frac{16}{3}\right)^2$$

$$\Rightarrow k = \frac{108}{3} - \left(\frac{16}{5}\right)^2$$

6. Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x + \frac{a}{x^2}\right)^n$, $x \neq 0$, be in the ratio 12 : 8 : 3. Then the term independent of x in the expansion, is equal to _______.

Official Ans. by NTA (4)

Sol. $T_{r+1} = {}^nC_r (x)^{n-r}\left(\frac{a}{x^2}\right)^r$

$= {}^nC_r\, a^r x^{n-3r}$

${}^nC_2\, a^2 : {}^nC_3\, a^3 : {}^nC_4\, a^4 = 12 : 8 : 3$

After solving

$n = 6,\ a = \frac{1}{2}$

For term independent of 'x' $\Rightarrow n = 3r$

$r = 2$

$\therefore$ Coefficient is ${}^6C_2\left(\frac{1}{2}\right)^2 = \frac{15}{4}$

Nearest integer is 4.

Ans. 4

7. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that AB = B and a + d = 2021, then the value of ad – bc is equal to ________.

Official Ans. by NTA (2020)

Sol. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

AB = B

$\Rightarrow$ (A – I) B = O

$\Rightarrow$ |A – I | = O, since B $\neq$ O

$$\begin{vmatrix} (a-1) & b \\ c & (d-1) \end{vmatrix} = 0$$

ad – bc = 2020

8. Let $\vec{x}$ be a vector in the plane containing vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$. If the vector $\vec{x}$ is perpendicular to $\left(3\hat{i} + 2\hat{j} - \hat{k}\right)$ and its projection on $\vec{a}$ is $\frac{17\sqrt{6}}{2}$, then the value of $|\vec{x}|^2$

CollegeDekho

Sol. Let $\vec{x} = \lambda\vec{a} + \mu\vec{b}$ (λ and μ are scalars)

$\vec{x} = \hat{i}(2\lambda+\mu) + \hat{j}(2\mu-\lambda) + \hat{k}(\lambda-\mu)$

Since $\vec{x}\cdot(3\hat{i}+2\hat{j}-\hat{k}) = 0$

$3\lambda + 8\mu = 0$ .....(1)

Also Projection of $\vec{x}$ on $\vec{a}$ is $\frac{17\sqrt{6}}{2}$

$$\frac{\vec{x}\cdot\vec{a}}{|\vec{a}|} = \frac{17\sqrt{6}}{2}$$

$6\lambda - \mu = 51$ .....(2)

From (1) and (2)

$\lambda = 8,\ \mu = -3$

$\vec{x} = 13\hat{i} - 14\hat{j} + 11\hat{k}$

$|\vec{x}|^2 = 486$ **Ans.**

9. Let $I_n = \int_1^e x^{19}\left(\log|x|\right)^n dx$, where $n \in N$. If $(20)I_{10} = \alpha I_9 + \beta I_8$, for natural numbers α and β, then $\alpha - \beta$ equal to _______.

Official Ans. by NTA (1)

Sol. $I_n = \int_1^e x^{19}\left(\log|x|\right)^n dx$

$$I_n = \left|\left(\log|x|\right)^{19}\frac{x^{20}}{20}\right|_1^e - \int n(\log|x|)^{n-1}\cdot\frac{1}{x}\cdot\frac{x^{20}}{20}dx$$

$20I_n = e^{20} - nI_{n-1}$

$\therefore\ 20I_{10} = e^{20} - 10I_9$

$20I_9 = e^{20} - 9I_8$

$\Rightarrow 20I_{10} = 10I_9 + 9I_8$

$\alpha = 10,\ \beta = 9$

10. Let P be an arbitrary point having sum of the squares of the distance from the planes $x + y + z = 0$, $lx - nz = 0$ and $x - 2y + z = 0$, equal to 9. If the locus of the point P is $x^2 + y^2 + z^2 = 9$, then the value of $l - n$ is equal to _______.

Official Ans. by NTA (0)

Sol. Let point P is (α, β, γ)

$$\left(\frac{\alpha+\beta+\gamma}{\sqrt{3}}\right)^2 + \left(\frac{\ell\alpha - n\gamma}{\sqrt{\ell^2+n^2}}\right)^2 + \left(\frac{\alpha-2\beta+\gamma}{\sqrt{6}}\right)^2 = 9$$

Locus is

$$\frac{(x+y+z)^2}{3} + \frac{(\ell x - nz)^2}{\ell^2+n^2} + \frac{(x-2y+z)^2}{6} = 9$$

$$x^2\left(\frac{1}{2}+\frac{\ell^2}{\ell^2+n^2}\right) + y^2 + z^2\left(\frac{1}{2}+\frac{n^2}{\ell^2+n^2}\right) + 2zx\left(\frac{1}{2}-\frac{\ell n}{\ell^2+n^2}\right) - 9 = 0$$

Since its given that $x^2 + y^2 + z^2 = 9$

After solving $\ell = n$