FINAL JEE–MAIN EXAMINATION – MARCH, 2021

(Held On Thursday 18th March, 2021) TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let y = y(x) be the solution of the differential equation $\frac{dy}{dx}=(y+1)\left((y+1)e^{x^2/2}-x\right), 0<x<2.1,$ with y(2) = 0. Then the value of $\frac{dy}{dx}$ at x = 1 is equal to :

(1) $\frac{-e^{3/2}}{(e^2+1)^2}$ (2) $-\frac{2e^2}{(1+e^2)^2}$

(3) $\frac{e^{5/2}}{(1+e^2)^2}$ (4) $\frac{5e^{1/2}}{(e^2+1)^2}$

Official Ans. by NTA (1)

Sol. Let y + 1 = Y

$\therefore \frac{dY}{dx} = Y^2 e^{\frac{x^2}{2}} - xY$

Put $-\frac{1}{Y}=k$

$\Rightarrow \frac{dk}{dx}+k(-x)=e^{\frac{x^2}{2}}$

I.F. = $e^{-\frac{x^2}{2}}$

$\therefore k=(x+c)e^{x^2/2}$

Put $k=-\frac{1}{y+1}$

$\therefore y+1=-\frac{1}{(x+c)e^{x^2/2}}$ ...(i)

when x = 2, y = 0, then c = –2 – $\frac{1}{e^2}$

diffentiate equation (i) & put x = 1

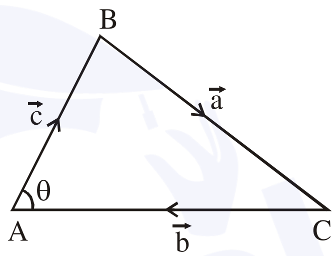
$\left(\frac{dy}{dx}\right) \quad \frac{e^{3/2}}{}$

2. In a triangle ABC, if $|\overrightarrow{BC}|=8$, $|\overrightarrow{CA}|=7$, $|\overrightarrow{AB}|=10$, then the projection of the vector $\overrightarrow{AB}$ on $\overrightarrow{AC}$ is equal to :

(1) $\frac{25}{4}$ (2) $\frac{85}{14}$ (3) $\frac{127}{20}$ (4) $\frac{115}{16}$

Official Ans. by NTA (2)

Sol.



$|\vec{a}|=8, |\vec{b}|=7, |\vec{c}|=10$

$\cos\theta=\frac{|\vec{b}|^2+|\vec{c}|^2-|\vec{a}|^2}{2|\vec{b}||\vec{c}|}=\frac{17}{28}$

Projection of $\vec{c}$ on $\vec{b}$

$=|\vec{c}|\cos\theta$

$=10\times\frac{17}{28}$

$=\frac{85}{14}$

3. Let the system of linear equations

4x + λy + 2z = 0

2x – y + z = 0

μx + 2y + 3z = 0, λ, μ ∈ R.

has a non-trivial solution. Then which of the following is true ?

(1) μ = 6, λ ∈ R (2) λ = 2, μ ∈ R

(3) λ = 3, μ ∈ R (4) μ = –6, λ ∈ R

Sol. For non-trivial solution

$$\begin{vmatrix} 4 & \lambda & 2 \\ 2 & -1 & 1 \\ \mu & 2 & 3 \end{vmatrix} = 0$$

$\Rightarrow 2\mu - 6\lambda + \lambda\mu = 12$

when $\mu = 6$, $12 - 6\lambda + 6\lambda = 12$

which is satisfied by all λ

4. Let $f : R - \{3\} \rightarrow R - \{1\}$ be defined by $f(x) = \dfrac{x-2}{x-3}$. Let $g : R \rightarrow R$ be given as $g(x) = 2x - 3$. Then, the sum of all the values of x for which $f^{-1}(x) + g^{-1}(x) = \dfrac{13}{2}$ is equal to

(1) 7 (2) 2 (3) 5 (4) 3

Official Ans. by NTA (3)

Sol. $f(x) = y = \dfrac{x-2}{x-3}$

$\therefore\ x = \dfrac{3y-2}{y-1}$

$\therefore\ f^{-1}(x) = \dfrac{3x-2}{x-1}$

& $g(x) = y = 2x - 3$

$\therefore\ x = \dfrac{y+3}{2}$

$\therefore\ g^{-1}(x) = \dfrac{x+3}{2}$

$\because\ f^{-1}(x) + g^{-1}(x) = \dfrac{13}{2}$

$\therefore\ x^2 - 5x + 6 = 0$ → x_1, x_2

$\therefore$ sum of roots

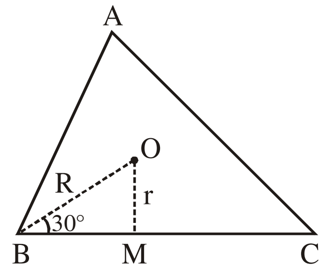
$x_1 + x_2 = 5$

5. Let the centroid of an equilateral triangle ABC be at the origin. Let one of the sides of the equilateral triangle be along the straight line $x + y = 3$. If R and r be the radius of circumcircle and incircle respectively of ΔABC, then $(R + r)$

(1) $\dfrac{9}{\sqrt{2}}$ (2) $7\sqrt{2}$ (3) $2\sqrt{2}$ (4) $3\sqrt{2}$

Official Ans. by NTA (1)

Sol.



$r = OM = \dfrac{3}{\sqrt{2}}$

& $\sin 30° = \dfrac{1}{2} = \dfrac{r}{R} \Rightarrow R = \dfrac{6}{\sqrt{2}}$

$\therefore r + R = \dfrac{9}{\sqrt{2}}$

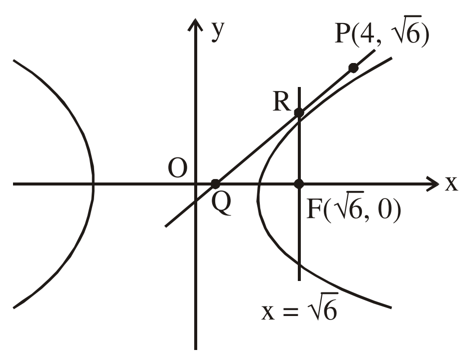
6. Consider a hyperbola $H : x^2 - 2y^2 = 4$. Let the tangent at a point $P(4, \sqrt{6})$ meet the x-axis at Q and latus rectum at $R(x_1, y_1)$, $x_1 > 0$. If F is a focus of H which is nearer to the point P, then the area of ΔQFR is equal to

(1) $4\sqrt{6}$ (2) $\sqrt{6} - 1$

(3) $\dfrac{7}{\sqrt{6}} - 2$ (4) $4\sqrt{6} - 1$

Official Ans. by NTA (3)

Sol.



$$\frac{x^2}{4} - \frac{y^2}{2} = 1$$

$$e = \sqrt{1 + \frac{b^2}{2}} = \sqrt{\frac{3}{}}$$

$\therefore$ Focus F(ae, 0) $\Rightarrow$ $F\left(\sqrt{6},0\right)$

equation of tangent at P to the hyperbola is

$2x - y\sqrt{6} = 2$

tangent meet x-axis at Q(1, 0)

& latus rectum x = $\sqrt{6}$ at $R\left(\sqrt{6},\frac{2}{\sqrt{6}}\left(\sqrt{6}-1\right)\right)$

$\therefore$ Area of $\Delta_{QFR} = \frac{1}{2}\left(\sqrt{6}-1\right).\frac{2}{\sqrt{6}}\left(\sqrt{6}-1\right)$

$=\frac{7}{\sqrt{6}}-2$

7. If P and Q are two statements, then which of the following compound statement is a tautology ?

(1) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow Q$

(2) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$

(3) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow P$

(4) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow (P \wedge Q)$

Official Ans. by NTA (2)

Sol. LHS of all the options are some i.e.

$((P \rightarrow Q) \wedge \sim Q)$

$\equiv (\sim P \vee Q) \wedge \sim Q$

$\equiv (\sim P \wedge \sim Q) \vee (Q \wedge \sim Q)$

$\equiv \sim P \wedge \sim Q$

(A) $(\sim P \wedge \sim Q) \rightarrow Q$

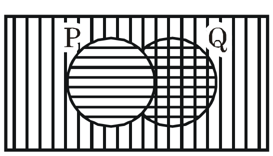
$\equiv \sim(\sim P \wedge \sim Q) \vee Q$

$\equiv (P \vee Q) \vee Q \neq$ tautology

(B) $(\sim P \wedge \sim Q) \rightarrow \sim P$

$\equiv \sim(\sim P \wedge \sim Q) \vee \sim P$

$\equiv (P \vee Q) \vee \sim P$



$\Rightarrow$ Tautology

(C) $(\sim P \wedge \sim Q) \rightarrow P$

$\equiv (P \vee Q) \vee P \neq$ Tautology

(D) $(\sim P \wedge \sim Q) \rightarrow (P \wedge Q)$

$\equiv (P \vee Q) \vee (P \wedge Q) \neq$ Tautology

Aliter :

P	Q	$P \vee Q$	$P \vee Q$	$\sim P$	$(P \vee Q) \vee \sim P$
T	T	T	T	F	T
T	F	T	F	F	T
F	T	T	F	T	T
F	F	F	F	T	T

8. Let $g(x)=\int_0^x f(t)\,dt$, where f is continuous function in [0, 3] such that $\frac{1}{3}\le f(t)\le 1$ for all $t \in [0, 1]$ and $0 \le f(t) \le \frac{1}{2}$ for all $t \in (1, 3]$. The largest possible interval in which g(3) lies is :

(1) $\left[-1,-\frac{1}{2}\right]$ (2) $\left[-\frac{3}{2},-1\right]$

(3) $\left[\frac{1}{3},2\right]$ (4) [1, 3]

Official Ans. by NTA (3)

Sol. $\frac{1}{3}\le f(t)\le 1 \;\forall\; t\in[0,1]$

$0\le f(t)\le\frac{1}{2} \;\forall\; t\in(1,3]$

Now, $g(3)=\int_0^3 f(t)\,dt=\int_0^1 f(t)\,dt+\int_1^3 f(t)\,dt$

$\because \int_0^1 \frac{1}{3}\,dt \le \int_0^1 f(t)\,dt \le \int_0^1 1.\,dt$ ....(1)

and $\int_1^3 0\,dt \le \int_1^3 f(1)\,dt \le \int_1^3 \frac{1}{2}\,dt$ ....(2)

Adding, we get

$\frac{1}{3}+0\le g(3)\le 1+\frac{1}{2}(3-1)$

$\frac{1}{\ }\le g(3)\le 2$

9. Let S_1 be the sum of first 2n terms of an arithmetic progression. Let S_2 be the sum of first 4n terms of the same arithmetic progression. If $(S_2 - S_1)$ is 1000, then the sum of the first 6n terms of the arithmetic progression is equal to:

(1) 1000 (2) 7000 (3) 5000 (4) 3000

Official Ans. by NTA (4)

Sol. $S_{2n} = \frac{2n}{2}[2a + (2n-1)d]$, $S_{4n} = \frac{4n}{2}[2a + (4n-1)d]$

$\Rightarrow S_2 - S_1 = \frac{4n}{2}[2a + (4n-1)d] - \frac{2n}{2}[2a + (2n-1)d]$

$= 4an + (4n-1)2nd - 2na - (2n-1)dn$

$= 2na + nd[8n - 2 - 2n + 1]$

$\Rightarrow 2na + nd[6n - 1] = 1000$

$2a + (6n-1)d = \frac{1000}{n}$

Now, $S_{6n} = \frac{6n}{2}\left[2a + (6n-1)d\right]$

$= 3n.\frac{1000}{n} = 3000$

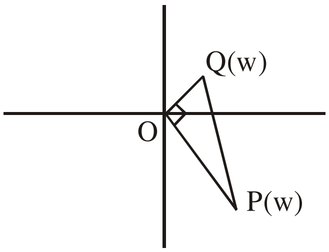
10. Let a complex number be $w = 1 - \sqrt{3}i$. Let another complex number z be such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and w is equal to :

(1) 4 (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) 2

Official Ans. by NTA (2)

Sol. $w = 1 - \sqrt{3}.i \Rightarrow |w| = 2$

Now, $|z| = \frac{1}{|w|} \Rightarrow |z| = \frac{1}{2}$



$\Rightarrow$ Area of triangle $= \frac{1}{2}.OP.OQ$

$= \frac{1}{2}.2.\frac{1}{2} = \frac{1}{2}$

11. Let in a series of 2n observations, half of them are equal to a and remaining half are equal to –a. Also by adding a constant b in each of these observations, the mean and standard deviation of new set become 5 and 20, respectively. Then the value of $a^2 + b^2$ is equal to :

(1) 425 (2) 650 (3) 250 (4) 925

Official Ans. by NTA (1)

Sol. Let observations are denoted by x_i for $1 \le i < 2n$

$\bar{x} = \frac{\sum x_i}{2n} = \frac{(a + a + ... + a) - (a + a + ... + a)}{2n}$

$\Rightarrow \bar{x} = 0$

and $\sigma_x^2 = \frac{\sum x_i^2}{2n} - (\bar{x})^2 = \frac{a^2 + a^2 + ... + a^2}{2n} - 0 = a^2$

$\Rightarrow \sigma_x = a$

Now, adding a constant b then $\bar{y} = \bar{x} + b = 5$

$\Rightarrow b = 5$

and $\sigma_y = \sigma_x$ (No change in S.D.) $\Rightarrow a = 20$

$\Rightarrow a^2 + b^2 = 425$

12. Let $S_1 : x^2 + y^2 = 9$ and $S_2 : (x-2)^2 + y^2 = 1$. Then the locus of center of a variable circle S which touches S_1 internally and S_2 externally always passes through the points :

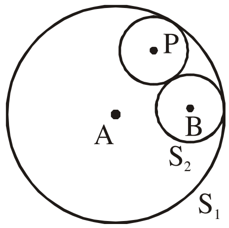
(1) $\left(0, \pm\sqrt{3}\right)$ (2) $\left(\frac{1}{2}, \pm\frac{\sqrt{5}}{2}\right)$

(3) $\left(2, \pm\frac{3}{2}\right)$ (4) $\left(1, \pm 2\right)$

Sol. $S_1 : x^2 + y^2 = 9$ — $r_1 = 3$, $A(0, 0)$

$S_2 : (x - 2)^2 + y^2 = 1$ — $r_2 = 1$, $B(2, 0)$

$\because c_1c_2 = r_1 - r_2$



$\therefore$ given circle are touching internally

Let a veriable circle with centre P and radius r

$\Rightarrow PA = r_1 - r$ and $PB = r_2 + r$

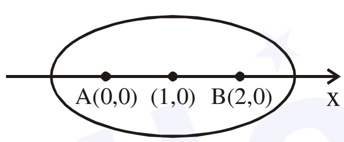
$\Rightarrow PA + PB = r_1 + r_2$

$\Rightarrow PA + PB = 4(> AB)$

$\Rightarrow$ Locus of P is an ellipse with foci at A(0, 0) and B(2, 0) and length of major axis is 2a = 4,

$e = \frac{1}{2}$

$\Rightarrow$ centre is at (1, 0) and $b^2 = a^2(1 - e^2) = 3$ if x-ellipse



$$\Rightarrow E: \frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$$

which is satisfied by $\left(2, \pm\frac{3}{2}\right)$

13. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors perpendicular to each other and $|\vec{a}| = |\vec{b}|$. If $|\vec{a} \times \vec{b}| = |\vec{a}|$, then the angle between the vectors $\left(\vec{a} + \vec{b} + \left(\vec{a} \times \vec{b}\right)\right)$ and $\vec{a}$ is equal to :

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(3) $\cos^{-1}\left(\frac{1}{\sqrt{\ }}\right)$ (4) $\sin^{-1}\left(\frac{1}{\sqrt{\ }}\right)$

Official Ans. by NTA (2)

Sol. $|\vec{a}| = |\vec{b}|$, $|\vec{a} \times \vec{b}| = |\vec{a}|$, $\vec{a} \perp \vec{b}$

$|\vec{a} \times \vec{b}| = |\vec{a}| \Rightarrow |\vec{a}||\vec{b}|\sin 90° = |\vec{a}| \Rightarrow |\vec{b}| = 1 = |\vec{a}|$

$\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors.

Let $\vec{a} = \hat{i}$, $\vec{b} = \hat{j} \Rightarrow \vec{a} \times \vec{b} = \hat{k}$

$$\cos\theta = \frac{\left(\hat{i} + \hat{j} + \hat{k}\right).\hat{i}}{\sqrt{3}\sqrt{1}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

14. Let in a Binomial distribution, consisting of 5 independent trials, probabilities of exactly 1 and 2 successes be 0.4096 and 0.2048 respectively. Then the probability of getting exactly 3 successes is equal to :

(1) $\frac{32}{625}$ (2) $\frac{80}{243}$ (3) $\frac{40}{243}$ (4) $\frac{128}{625}$

Official Ans. by NTA (1)

Sol. $P(X = 1) = {}^5C_1.p.q^4 = 0.4096$

$P(X = 2) = {}^5C_2.p^2.q^3 = 0.2048$

$\Rightarrow \frac{q}{2p} = 2$

$\Rightarrow q = 4p$ and $p + q = 1$

$\Rightarrow p = \frac{1}{5}$ and $q = \frac{4}{5}$

Now

$$P(X = 3) = {}^5C_3.\left(\frac{1}{5}\right)^3.\left(\frac{4}{5}\right)^2 = \frac{10 \times 16}{125 \times 25} = \frac{32}{625}$$

15. Let a tangent be drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $\left(3\sqrt{3}\cos\theta, \sin\theta\right)$ where $\theta \in \left(0, \frac{\pi}{2}\right)$. Then the value of θ such that the sum of intercepts on axes made by this tangent is minimum is equal to :

(1) $\frac{\pi}{8}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{3}$

Sol. Equation of tangent be

$$\frac{x\cos\theta}{3\sqrt{3}}+\frac{y.\sin\theta}{1}=1,\qquad \theta\in\left(0,\frac{\pi}{2}\right)$$

intercept on x-axis

$OA = 3\sqrt{3}\sec\theta$

intercept on y-axis

$OB = \text{cosec}\theta$

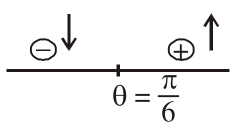
Now, sum of intercept

$= 3\sqrt{3}\sec\theta + \text{cosec}\theta = f(\theta)$ let

$f'(\theta) = 3\sqrt{3}\sec\theta\tan\theta - \text{cosec}\theta\cot\theta$

$$=3\sqrt{3}\frac{\sin\theta}{\cos^2\theta}-\frac{\cos\theta}{\sin^2\theta}$$

$$=\underbrace{\frac{\cos\theta}{\sin^2\theta}.3\sqrt{3}}_{\oplus}\left[\tan^3\theta-\frac{1}{3\sqrt{3}}\right]=0\Rightarrow\theta=\frac{\pi}{6}$$



$\Rightarrow$ at $\theta = \frac{\pi}{6}$, $f(\theta)$ is minimum

16. Define a relation R over a class of n × n real matrices A and B as "ARB iff there exists a non-singular matrix P such that $PAP^{-1} = B$". Then which of the following is true ?

(1) R is symmetric, transitive but not reflexive,
(2) R is reflexive, symmetric but not transitive
(3) R is an equivalence relation
(4) R is reflexive, transitive but not symmetric

Official Ans. by NTA (3)

Sol. A and B are matrices of n × n order & ARB iff there exists a non singular matrix P(det(P) ≠ 0) such that $PAP^{-1} = B$

For reflexive

$ARA \Rightarrow PAP^{-1} = A$ ...(1) must be true

for P = I, Eq.(1) is true so 'R' is reflexive

For symmetric

$ARB \Leftrightarrow PAP^{-1} = B$ ...(1) is true

for BRA iff $PBP^{-1} = A$ ...(2) must be true

$\because PAP^{-1} = B$

$P^{-1}PAP^{-1} = P^{-1}B$

$IAP^{-1}P = P^{-1}BP$

from (2) & (3) $PBP^{-1} = P^{-1}BP$

can be true some $P = P^{-1} \Rightarrow P^2 = I$ (det(P) ≠ 0)

So 'R' is symmetric

For trnasitive

$ARB \Leftrightarrow PAP^{-1} = B$... is true

$BRC \Leftrightarrow PBP^{-1} = C$... is true

now $PPAP^{-1}P^{-1} = C$

$P^2A(P^2)^{-1} = C \Rightarrow ARC$

So 'R' is transitive relation

$\Rightarrow$ Hence R is equivalence

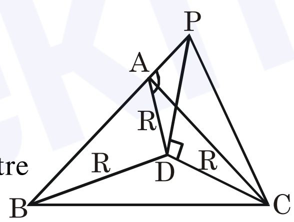
17. A pole stands vertically inside a triangular park ABC. Let the angle of elevation of the top of the pole from each corner of the park be $\frac{\pi}{3}$. If the radius of the circumcircle ot ΔABC is 2, then the height of the pole is equal to :

(1) $\frac{2\sqrt{3}}{3}$ (2) $2\sqrt{3}$ (3) $\sqrt{3}$ (4) $\frac{1}{\sqrt{3}}$

Official Ans. by NTA (2)

Sol. Let PD = h, R = 2

As angle of elevation of top of pole from A, B, C are equal So D must be circumcentre of ΔABC



$$\tan\left(\frac{\pi}{3}\right)=\frac{PD}{R}=\frac{h}{R}$$

$$h=R\tan\left(\frac{\pi}{3}\right)=2\sqrt{3}$$

18. If $15\sin^4\alpha + 10\cos^4\alpha = 6$, for some α ∈ R, then the value of $27\sec^6\alpha + 8\text{cosec}^6\alpha$ is equal to :

(1) 350 (2) 500 (3) 400 (4) 250

Official Ans. by NTA (4)

Sol. $15\sin^4\alpha + 10\cos^4\alpha = 6$

$15\sin^4\alpha + 10\cos^4\alpha = 6(\sin^2\alpha + \cos^2\alpha)^2$

$(3\sin^2\alpha - 2\cos^2\alpha)^2 = 0$

$$\tan^2\alpha=\frac{2}{3}.\ \cot^2\alpha=\frac{3}{2}$$

$\Rightarrow 27\sec^6\alpha + 8\text{cosec}^6\alpha$

$= 27(\sec^6\alpha)^3 + 8(\text{cosec}^6\alpha)^3$

$= 27(1 + \tan^2\alpha)3 + 8(1 + \cot^2\alpha)^3$

19. The area bounded by the curve $4y^2 = x^2(4 - x)(x - 2)$ is equal to :

(1) $\frac{\pi}{8}$ (2) $\frac{3\pi}{8}$ (3) $\frac{3\pi}{2}$ (4) $\frac{\pi}{16}$

Official Ans. by NTA (3)

Sol. $4y^2 = x^2(4 - x)(x - 2)$

$$|y| = \frac{|x|}{2}\sqrt{(4-x)(x-2)}$$

$$\Rightarrow y_1 = \frac{x}{2}\sqrt{(4-x)(x-2)}$$

$$\text{and } y_2 = \frac{-x}{2}\sqrt{(4-x)(x-2)}$$

$D : x \in [2, 4]$

Required Area

$$= \int_2^4 (y_1 - y_2)\,dx = \int_2^4 x\sqrt{(4-x)(x-2)}\,dx \quad \ldots(1)$$

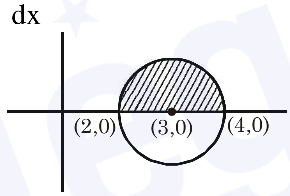
$$\text{Applying } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$\text{Area} = \int_2^4 (6-x)\sqrt{(4-x)(x-2)}\,dx \quad \ldots(2)$$

(1) + (2)

$$2A = 6\int_2^4 \sqrt{(4-x)(x-2)}\,dx$$

$$A = 3\int_2^4 \sqrt{1-(x-3)^2}\,dx$$



$$A = 3.\frac{\pi}{2}.1^2 = \frac{3\pi}{2}$$

20. Let $f : R \to R$ be a function defined as

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin 2x}{2x} & , \text{ if } x < 0 \\ b & , \text{ if } x = 0 \\ \dfrac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} & , \text{ if } x > 0 \end{cases}$$

If f is continuous at $x = 0$, then the value of $a + b$ is equal to :

(1) $-\frac{5}{2}$ (2) -2 (3) -3 (4) $-\frac{3}{2}$

Official Ans. by NTA (4)

Sol. $f(x)$ is continuous at $x = 0$

$$\lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x) \quad \ldots(1)$$

$$f(0) = b \quad \ldots(2)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{\sin(a+1)x}{2x} + \frac{\sin 2x}{2x}\right)$$

$$= \frac{a+1}{2} + 1 \quad \ldots(3)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}}$$

$$= \lim_{x\to 0^+} \frac{(x+bx^3-x)}{bx^{5/2}\left(\sqrt{x+bx^3}+\sqrt{x}\right)}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1+bx^2}+1\right)} = \frac{1}{2} \quad \ldots(4)$$

Use (2), (3) & (4) in (1)

$$\frac{1}{2} = b = \frac{a+1}{2} + 1$$

$$\Rightarrow b = \frac{1}{2},\ a = -2$$

$$a + b = \frac{-3}{2}$$

SECTION-B

1. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $P(x) = f(x^3) + xg(x^3)$ is divisible by $x^2 + x + 1$, then P(l) is equal to______.

Official Ans. by NTA (0)

Sol. $P(x) = f(x^3) + xg(x^3)$

$P(1) = f(1) + g(1)$ ...(1)

Now $P(x)$ is divisible by $x^2 + x + 1$

$\Rightarrow P(x) = Q(x)(x^2 + x + 1)$

$P(w) = 0 = P(w^2)$ where w, w^2 are non-real cube roots of units

$P(x) = f(x^3) + xg(x^3)$

$P(w) = f(w^3) + wg(w^3) = 0$

$f(1) + wg(1) = 2$ ...(2)

$P(w^2) = f(w^6) + w^2g(w^6) = 0$

(2) + (3)

$\Rightarrow 2f(1) + (w + w^2)g(1) = 0$

$2f(1) = g(1)$ ...(4)

(2) – (3)

$\Rightarrow (w - w^2)g(1) = 0$

$g(1) = 0 = f(1)$ from (4)

from (1) $P(1) = f(1) + g(1) = 0$

2. Let I be an identity matrix of order 2 × 2 and $P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$. Then the value of $n \in N$ for which $P^n = 5I - 8P$ is equal to ______.

Official Ans. by NTA (6)

Sol. $P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$

$$5I - 8P = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 16 & -8 \\ 40 & -24 \end{bmatrix} = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix} \Rightarrow P^6 = \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix} = P^n$$

$\Rightarrow n = 6$

3. If $\sum_{r=1}^{10} r!\left(r^3 + 6r^2 + 2r + 5\right) = \alpha(11!)$, then the value of α is equal to ______.

Official Ans. by NTA (160)

Sol. $\sum_{r=1}^{10} r!\{(r+1)(r+2)(r+3) - 9(r+1) + 8\}$

$$= \sum_{r=1}^{10}\left[\{(r+3)! - (r+1)!\} - 8\{(r+1)! - r!\}\right]$$

$= (13! + 12! - 2! - 3!) - 8(11! - 1)$

$= (12.13 + 12 - 8).11! - 8 + 8$

$= (160)(11)!$

Hence $\alpha = 160$

4. The term independent of x in the expansion of $\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right]^{10}$, $x \neq 1$, is equal to ______.

Official Ans. by NTA (210)

Sol. $\left(\left(x^{1/3} + 1\right) - \left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10}$

$(x^{1/3} - x^{-1/2})^{10}$

$T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r$

$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 2r - 3r = 0$

$\Rightarrow r = 4$

$$T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

5. Let P(x) be a real polynomial of degree 3 which vanishes at x = –3. Let P(x) have local minima at x = 1, local maxima at x = –1 and $\int_{-1}^{1} P(x)\,dx = 18$, then the sum of all the coefficients of the polynomial P(x) is equal to ______.

Official Ans. by NTA (8)

Sol. Let $p'(x) = a(x - 1)(x + 1) = a(x^2 - 1)$

$p(x) = a\int\left(x^2 - 1\right)dx + c$

$= a\left(\frac{x^3}{3} - x\right) + c$

Now $p(-3) = 0$

$\Rightarrow a\left(-\frac{27}{3} + 3\right) + c = 0$

$\Rightarrow -6a + c = 0$ ...(1)

Now $\int_{-1}^{1}\left(a\left(\frac{x^3}{3} - x\right) + c\right)dx = 18$

$= 2c = 18 \Rightarrow c = 9$ ...(2)

$\Rightarrow p(x) = \frac{3}{2}\left(\frac{x^3}{3} - x\right) + 9$

sum of coefficient

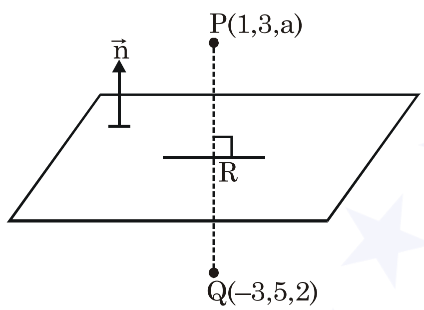
$= \frac{1}{2} - \frac{3}{2} + 9$

$= 8$

6. Let the mirror image of the point (1, 3, a) with respect to the plane $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) - b = 0$ be (–3, 5, 2). Then the value of |a + b| is equal to _______.

Official Ans. by NTA (1)

Sol.



plane = 2x – y + z = b

$R \equiv \left(-1, 4, \frac{a+2}{2}\right) \rightarrow$ on plane

$\therefore -2 - 4 + \frac{a+2}{2} = b$

$\Rightarrow a + 2 = 2b + 12 \Rightarrow a = 2b + 10$ ...(i)

<PQ> = <4, –2, a – 2>

$\therefore \frac{2}{4} = \frac{-1}{-2} = \frac{1}{a-2}$

$\Rightarrow a - 2 = 2 \Rightarrow a = 4,\ b = -3$

$\therefore |a + b| = 1$

7. Let f : R → R satisfy the equation $f(x + y) = f(x).f(y)$ for all x, y ∈ R and f(x) ≠ 0 for any x ∈ R. If Ihe function f is differentiable at x = 0 and $f'(0) = 3$, then

$\lim_{h \to 0} \frac{1}{h}(f(h) - 1)$ is equal to ______.

Official Ans. by NTA (3)

Sol. If $f(x + y) = f(x).f(y)$ & $f'(0) = 3$ then

$f(x) = a^x \Rightarrow f'(x) = a^x.\ell na$

$\Rightarrow f'(0) = \ell na = 3 \Rightarrow a = e^3$

$\Rightarrow f(x) = (e^3)^x = e^{3x}$

$$\lim_{x \to 0} \frac{f(x) - 1}{x} = \lim_{x \to 0}\left(\frac{e^{3x} - 1}{3x} \times 3\right) = 1 \times 3 = 3$$

8. Let ${}^nC_r$ denote the binomial coefficient of x^r in the expansion of $(1 + x)^n$.

If $\sum_{k=0}^{10}(2^2 + 3k)\,{}^nC_k = \alpha.3^{10} + \beta.2^{10}$, α, β ∈ R, then α + β is equal to ______.

Official Ans. by NTA (19)

Answer (Bonus)

Sol. **BONUS**

Instead of ${}^nC_k$ it must be ${}^{10}C_k$ i.e.

$$\sum_{k=0}^{10}(2^2 + 3k)\,{}^{10}C_k = \alpha.3^{10} + \beta.2^{10}$$

$$\text{LHS} = 4\sum_{k=0}^{10} {}^{10}C_k + 3\sum_{k=0}^{10} k.\frac{10}{k}.{}^9C_{k-1}$$

$= 4.2^{10} + 3.10.2^9$

$= 19.2^{10} = \alpha.3^{10} + \beta.2^{10}$

$\Rightarrow \alpha = 0,\ \beta = 19 \Rightarrow \alpha + \beta = 19$

9. Let P be a plane containing the line $\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+5}{2}$ and parallel to the line $\frac{x-3}{4} = \frac{y-2}{-3} = \frac{z+5}{7}$. If the point (1, –1, α) lies on the plane P, then the value of |5α| is equal to ______.

Official Ans. by NTA (38)

Sol. Equation of plane is $\begin{vmatrix} x-1 & y+6 & z+5 \\ 3 & 4 & 2 \\ 4 & -3 & 7 \end{vmatrix} = 0$

Now (1, –1, α) lies on it so

$$\begin{vmatrix} 0 & 5 & \alpha+5 \\ 3 & 4 & 2 \\ 4 & -3 & 7 \end{vmatrix} = 0 \Rightarrow 5\alpha + 38 = 0 \Rightarrow |5\alpha| = 38$$

10. Let y = y(x) be the solution of the differential equation $xdy - ydx = \sqrt{(x^2 - y^2)}\,dx$, $x \geq 1$, with $y(1) = 0$. If the area bounded by the line $x = 1$, $x = e^{\pi}$, $y = 0$ and $y = y(x)$ is $\alpha e^{2\pi} + \beta$, then the value of $10(\alpha + \beta)$ is equal to _______.

Official Ans. by NTA (4)

Sol. $xdy - ydx = \sqrt{x^2 - y^2}\,dx$

$$\Rightarrow \frac{xdy - ydx}{x^2} = \frac{1}{x}\sqrt{1 - \frac{y^2}{x^2}}\,dx$$

$$\Rightarrow \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \ell n|x| + c$$

at $x = 1$, $y = 0 \Rightarrow c = 0$

$y = x\sin(\ell nx)$

$$A = \int_1^{e^{\pi}} x\sin(\ell n x)\,dx$$

$x = e^t$, $dx = e^t dt \Rightarrow \int_0^{\pi} e^{2t}\sin(t)\,dt = A$

$$\alpha e^{2\pi} + \beta = \left(\frac{e^{2t}}{5}(2\sin t - \cos t)\right)_0^{\pi} = \frac{1 + e^{2\pi}}{5}$$

$\alpha = \frac{1}{5}, \beta = \frac{1}{5}$ so $10(\alpha + \beta) = 4$