MATHEMATICS

SECTION-A

1. For the natural numbers m, n, if

$(1-y)^m (1+y)^n = 1 + a_1 y + a_2 y^2 + ... + a_{m+n} y^{m+n}$

and $a_1 = a_2 = 10$, then the value of $(m+n)$ is equal to :

(1) 88 (2) 64

(3) 100 (4) 80

Official Ans. by NTA (4)

Sol. $(1-y)^m (1+y)^n$

Coefficient of y $(a_1) = 1.{}^nC_1 + {}^mC_1(-1)$

$= n - m = 10$ .....(1)

Coefficient of y^2 (a_2)

$= 1.{}^nC_2 - {}^mC_1.{}^nC_1. + 1.{}^mC_2 = 10$

$= \frac{n(n-1)}{2} - m.n + \frac{m(m-1)}{2} = 10$

$m^2 + n^2 - 2mn - (n+m) = 20$

$(n-m)^2 - (n+m) = 20$

$n + m = 80$ ..... (2)

By equation (1) & (2)

$m = 35, n = 45$

2. The value of $\tan\left(2\tan^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)\right)$ is equal to :

(1) $\frac{-181}{69}$ (2) $\frac{220}{21}$ (3) $\frac{-291}{76}$ (4) $\frac{151}{63}$

Official Ans. by NTA (2)

Sol. $\underbrace{\tan^{-1}\frac{3}{5}+\tan^{-1}\frac{3}{5}}_{x>0,y>0,xy<1}+\tan^{-1}\frac{5}{12}$

$\tan^{-1}\frac{\frac{6}{5}}{1-\frac{9}{25}} = \underbrace{\tan^{-1}\frac{15}{8}+\tan^{-1}\frac{5}{12}}_{x>0,y>0,xy<1}$

$\tan^{-1}\frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{15}{8}\cdot\frac{5}{12}} = \tan^{-1}\frac{220}{21}$

$\tan\left(\tan^{-1}\frac{220}{21}\right) = \frac{220}{21}$

3. Let r_1 and r_2 be the radii of the largest and smallest circles, respectively, which pass through the point $(-4,1)$ and having their centres on the circumference of the circle $x^2 + y^2 + 2x + 4y - 4 = 0$.

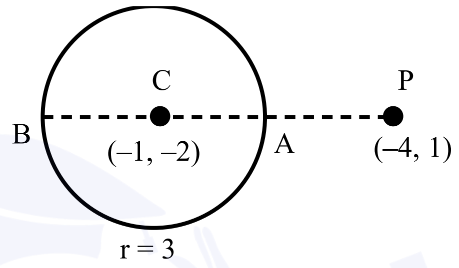
If $\frac{r_1}{r_2} = a + b\sqrt{2}$, then a + b is equal to :

(1) 3 (2) 11

(3) 5 (4) 7

Official Ans. by NTA (3)

Sol.



Centre of smallest circle is A

Centre of largest circle is B

$r_2 = |CP - CA| = 3\sqrt{2} - 3$

$r_1 = CP + CB = 3\sqrt{2} + 3$

$\frac{r_1}{r_2} = \frac{3\sqrt{2}+3}{3\sqrt{2}-3} = \frac{(3\sqrt{2}+3)^2}{9} = (\sqrt{2}+1)^2 = 3 + 2\sqrt{2}$

$a = 3, b = 2$

4. Consider the following three statements :

(A) If 3 + 3 = 7 then 4 + 3 = 8.

(B) If 5 + 3 = 8 then earth is flat.

(C) If both (A) and (B) are true then 5 + 6 = 17.

Then, which of the following statements is correct ?

(1) (A) is false, but (B) and (C) are true

(2) (A) and (C) are true while (B) is false

(3) (A) is true while (B) and (C) are false

(4) (A) and (B) are false while (C) is true

Official Ans. by NTA (2)

Sol. Truth Table

P	q	P□q
T	T	T
T	F	F
F	T	T
F	F	T

5. The lines x = ay – 1 = z – 2 and

x = 3y – 2 = bz – 2, ($ab \neq 0$) are coplanar, if :

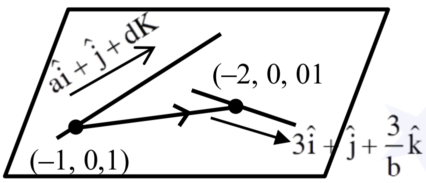
(1) $b = 1,\ a \in R - \{0\}$ (2) $a = 1,\ b \in R - \{0\}$

(3) a = 2, b = 2 (4) a = 2, b = 3

Official Ans. by NTA (1)

Sol. $\frac{x+1}{a} = y = \frac{z-1}{a}$

$\frac{x+2}{3} = y = \frac{z}{3/b}$



lines are Co-planar

$$\begin{vmatrix} a & 1 & a \\ 3 & 1 & \frac{3}{b} \\ -1 & 0 & -1 \end{vmatrix} = 0 \Rightarrow -\left(\frac{3}{b} - a\right) - 1(a-3) = 0$$

$$a - \frac{3}{b} - a + 3 = 0$$

$$b = 1, a \in R - \{0\}$$

6. If [x] denotes the greatest integer less than or equal to x, then the value of the integral

$\int_{-\pi/2}^{\pi/2} [[x] - \sin x] dx$ is equal to :

(1) $-\pi$ (2) π (3) 0 (4) 1

Official Ans. by NTA (1)

Sol. $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ([x] + [-\sin x]) dx$ .....(i)

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ([-x] + [\sin x]) dx$ .....(2)

(King property)

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\underbrace{[x] + [-x]}_{-1} \right) + \left(\underbrace{[\sin x] + [-\sin x]}_{-1} \right) dx$$

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-2) dx = -2(\pi)$$

$$I = -\pi$$

7. If the real part of the complex number $(1 - \cos\theta + 2i\sin\theta)^{-1}$ is $\frac{1}{5}$ for $\theta \in (0, \pi)$, then the value of the integral $\int_0^\theta \sin x\, dx$ is equal to :

(1) 1 (2) 2 (3) –1 (4) 0

Official Ans. by NTA (1)

Sol. $z = \frac{1}{1 - \cos\theta + 2i\sin\theta}$

$$= \frac{2\sin^2\frac{\theta}{2} - 2i\sin\theta}{(1-\cos\theta)^2 + 4\sin^2\theta}$$

$$= \frac{\sin\frac{\theta}{2} - 2i\cos\frac{\theta}{2}}{4\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\right)}$$

$$\text{Re}(z) = \frac{1}{2\left(\sin^2\frac{\theta}{2} + 4\cos^2\frac{\theta}{2}\right)} = \frac{1}{5}$$

$$\sin\frac{2\theta}{2} + 4\cos^2\frac{\theta}{2} = \frac{5}{2}$$

$$1 - \cos^2\frac{\theta}{2} + 4\cos\frac{\theta}{2} = \frac{5}{2}$$

$$3\cos^2\frac{\theta}{2} = \frac{3}{2}$$

$$\cos^2\frac{\theta}{2} = \frac{1}{2}$$

$$\frac{\theta}{2} = n\pi \pm \frac{\pi}{4}$$

$$\theta = 2n\pi \pm \frac{\pi}{2}$$

$$\theta = 2n\pi \pm \frac{\pi}{2}$$

$\theta \in (0,\pi)$

$$\boxed{\theta = \frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta - [-\cos\theta]_0^{\frac{\pi}{2}}$$

$= -(0-1)$

$= 1$

8. Let $f : \mathbf{R} - \left\{\frac{\alpha}{6}\right\} \to \mathbf{R}$ be defined by $f(x) = \frac{5x+3}{6x-\alpha}$. Then the value of α for which $(fof)(x) = x$, for all $x \in \mathbf{R} - \left\{\frac{\alpha}{6}\right\}$, is :

(1) No such α exists (2) 5

(3) 8 (4) 6

Official Ans. by NTA (2)

Sol. $f(x) = \frac{5x+3}{6x-\alpha} = y$ ......(i)

$5x + 3 = 6xy - \alpha y$

$x(6y-5) = \alpha y + 3$

$x = \frac{\alpha y+3}{6y-5}$

$f^{-1}(x) = \frac{\alpha x+3}{6x-5}$ .....(ii)

fo $f(x) = x$

$f(x) = f^{-1}(x)$

From eq^n (i) & (ii)

Clearly $(\alpha = 5)$

9. If $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = x + 1$, then the value of

$$\lim_{n\to\infty} \frac{1}{n}\left[f(0) + f\left(\frac{5}{n}\right) + f\left(\frac{10}{n}\right) + ... + f\left(\frac{5(n-1)}{n}\right)\right],$$

is :

(1) $\frac{3}{2}$ (2) $\frac{5}{2}$

(3) $\frac{1}{2}$ (4) $\frac{7}{2}$

Official Ans. by NTA (4)

Sol. $I = \sum_{r=0}^{n-1} f\left(\frac{5r}{n}\right)\frac{1}{n}$

$I = \int_0^1 f(5x)dx$

$I = \int_0^1 (5x+1)dx$

$I = \left[\frac{5x^2}{2} + x\right]_0^1$

$I = \frac{5}{2} + 1 = \frac{7}{2}$

10. Let A, B and C be three events such that the probability that exactly one of A and B occurs is $(1-k)$, the probability that exactly one of B and C occurs is $(1-2k)$, the probability that exactly one of C and A occurs is $(1-k)$ and the probability of all A, B and C occur simultaneously is k^2, where $0 < k < 1$. Then the probability that at least one of A, B and C occur is :

(1) greater than $\frac{1}{8}$ but less than $\frac{1}{4}$

(2) greater than $\frac{1}{2}$

(3) greater than $\frac{1}{4}$ but less than $\frac{1}{2}$

(4) exactly equal to $\frac{1}{2}$

Official Ans. by NTA (2)

Sol. $P(\bar{A} \cap B) + P(A \cap \bar{B}) = 1 - k$

$P(\bar{A} \cap C) + P(A \cap \bar{C}) = 1 - 2k$

$P(\bar{B} \cap C) + P(B \cap \bar{C}) = 1 - k$

$P(A \cap B \cap C) = k^2$

$P(A) + P(B) - 2\,P(A \cap B) = 1 - k$ .....(i)

$P(B) + P(C) - 2P(B \cap C) = 1 - k$ .....(ii)

$P(C) + P(A) - 2P(A \cap C) = 1 - 2k$ .....(iii)

(1) + (2) + (3)

$P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)$

$-P(C \cap A) = \frac{-4k+3}{2}$

So,

$$P(A\cup B\cup C)=\frac{-4k+3}{2}+k^2$$

$$P(A\cup B\cup C)=\frac{2k^2-4k+3}{2}$$

$$=\frac{2(k-1)^2+1}{2}$$

$$P(A\cup B\cup C)>\frac{1}{2}$$

11. The sum of all the local minimum values of the twice differentiable function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x)=x^3-3x^2-\frac{3f''(2)}{2}x+f''(1)$ is :

(1) –22 (2) 5 (3) –27 (4) 0

Official Ans. by NTA (3)

Sol. $f(x) = x^3 - 3x^2 - \frac{3}{2}f''(2)\,x + f''(1)$ .....(i)

$f'(x) = 3x^2 - 6x - \frac{3}{2}f''(2)$ .....(ii)

$f''(x) = 6x - 6$ .....(iii)

Now is 3[rd] equation

$f''(2) = 12 - 6 = 6$

$f''(11 = 0)$

Use (ii)

$f'(x) = 3x^2 - 6x - \frac{3}{2}f''(2)$

$f'(x) = 3x^2 - 6x - \frac{3}{2}\times 6$

$f'(x) = 3x^2 - 6x - 9$

$f'(x) = 0$

$3x^2 - 6x - 9 = 0$

$\Rightarrow x = -1 \ \& \ 3$

Use (iii)

$f''(x) = 6x - 6$

$f''(-1) = -12 < 0$ maxima

$f''(3) = 12 > 0$ minima.

Use (i)

$f(x) = x^3 - 3x^2 - \frac{3}{2}\ f''(2)\ x + f''(1)$

$f(x) = x^3 - 3x^2 - \frac{3}{2} \times 6 \times x + 0$

$f(x) = x^3 - 3x^2 - 9x$

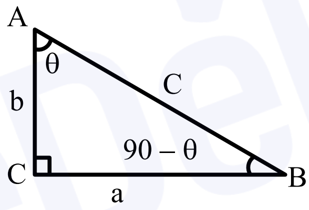
$f(3) = 27 - 27 - 9 \times 3 = -27$

12. Let in a right angled triangle, the smallest angle be θ. If a triangle formed by taking the reciprocal of its sides is also a right angled triangle, then sinθ is equal to :

(1) $\frac{\sqrt{5}+1}{4}$ (2) $\frac{\sqrt{5}-1}{2}$ (3) $\frac{\sqrt{2}-1}{2}$ (4) $\frac{\sqrt{5}-1}{4}$

Official Ans. by NTA (2)

Sol.



$\angle A = \theta$

$\angle B = 90 - \theta$

a = smallest side

$c^2 = a^2 + b^2$

$$\frac{1}{a^2}=\frac{1}{b^2}+\frac{1}{c^2}$$

$$\frac{b^2c^2}{a^2}=b^2+c^2$$

Use $a = 2R \sin A = 2R \sin\theta$

$b = 2R \sin B = 2R \sin (90 - \theta) = 2R \cos\theta$

$c = 2R \sin C = 2 \sin 90° = 2R$

$$\frac{4R^2\cos^2\theta}{4R^2\sin^2\theta}=4R^2\cos^2\theta+4R^2$$

$\cos^2\theta = \sin^2\theta\cos^2\theta + \sin^2\theta$

$1 - \sin^2\theta = \sin^2\theta(1 - \sin^2\theta) + \sin^2\theta$

$$\sin^2\theta=\frac{3-\sqrt{5}}{2}$$

$$\Rightarrow \sin\theta=\frac{\sqrt{5}-1}{2}$$

13. Let y=y(x) satisfies the equation $\frac{dy}{dx}-|A|=0$, for all x > 0, where $A=\begin{bmatrix} y & \sin x & 1 \\ 0 & -1 & 1 \\ 2 & 0 & \frac{1}{x}\end{bmatrix}$. If $y(\pi)=\pi+2$, then the value of $y\left(\frac{\pi}{2}\right)$ is :

(1) $\frac{\pi}{2}+\frac{4}{\pi}$ (2) $\frac{\pi}{2}-\frac{1}{\pi}$ (3) $\frac{3\pi}{2}-\frac{1}{\pi}$ (4) $\frac{\pi}{2}-\frac{4}{\pi}$

Official Ans. by NTA (1)

Sol. $|A|=-\frac{y}{x}+2\sin x+2$

$$\frac{dy}{dx}=|A|$$

$$\frac{dy}{dx}=-\frac{y}{x}+2\sin x+2$$

$$\frac{dy}{dx}+\frac{y}{x}=2\sin x+2$$

$$\text{I.F.}=e^{\int\frac{1}{x}dx}=x$$

$$\Rightarrow yx=\int x(2\sin x+2)dx$$

$xy = x^2 - 2x\cos x + 2\sin x + c$ .....(i)

Now $x = \pi, y = \pi + 2$

Use in (i)

$c = 0$

Now (i) be comes

$xy = x^2 - 2x\cos x + 2\sin x$

put $x = \pi/2$

$$\frac{\pi}{2}y=\left(\frac{\pi}{2}\right)^2-2.\frac{\pi}{2}\cos\frac{\pi}{2}+2\sin\frac{\pi}{2}$$

$$\frac{\pi}{2}y=\frac{\pi^2}{4}+2$$

14. Consider the line L given by the equation $\frac{x-3}{2}=\frac{y-1}{1}=\frac{z-2}{1}$. Let Q be the mirror image of the point (2, 3, –1) with respect to L. Let a plane P be such that it passes through Q, and the line L is perpendicular to P. Then which of the following points is on the plane P ?

(1) (–1, 1, 2) (2) (1, 1, 1)

(3) (1, 1, 2) (4) (1, 2, 2)

Official Ans. by NTA (4)

Sol. Plane p is $\perp^r$ to line

$$\frac{x-3}{2}=\frac{y-1}{1}=\frac{z-2}{1}$$

& passes through pt. (2, 3) equation of plane p

$2(x - 2) + 1(y - 3) + 1(z + 1) = 0$

$2x + y + z - 6 = 0$

pt (1,2,2) satisfies above equation

15. If the mean and variance of six observations 7, 10, 11, 15, a, b are 10 and $\frac{20}{3}$, respectively, then the value of $|a-b|$ is equal to :

(1) 9 (2) 11 (3) 7 (4) 1

Official Ans. by NTA (4)

Sol. $10=\frac{7+10+11+15+a+b}{6}$

$\Rightarrow a + b = 17$ .....(i)

$$\frac{20}{3}=\frac{7^2+10^2+11^2+15^2+a^2+b^2}{6}-10^2$$

$a^2 + b^2 = 145$ ......(ii)

Solve (i) and (ii) a = 9, b = 8 or a = 8, b = 9

$|a - b| = 1$

16. Let $g(t)=\int_{-\pi/2}^{\pi/2}\cos\left(\frac{\pi}{4}t+f(x)\right)dx$, where $f(x)=\log_e\left(x+\sqrt{x^2+1}\right)$, $x\in\mathbf{R}$. Then which one of the following is correct ?

(1) $g(1)=g(0)$ (2) $\sqrt{2}g(1)=g(0)$

(3) $g(1)=\sqrt{2}g(0)$ (4) $g(1)+g(0)=0$

Official Ans. by NTA (2)

Sol. $g(t)=\int_{-\pi/2}^{\pi/2}\left(\cos\frac{\pi}{4}t+f(x)\right)dx$

$g(t)=\pi\cos\frac{\pi}{4}t+\int_{-\pi/2}^{\pi/2}f(x)dx$

$g(t)=\pi\cos\frac{\pi}{4}t$

$g(1)=\frac{\pi}{\sqrt{2}},g(0)=\pi$

17. Let P be a variable point on the parabola $y = 4x^2 + 1$. Then, the locus of the mid-point of the point P and the foot of the perpendicular drawn from the point P to the line y = x is :

(1) $(3x - y)^2 + (x - 3y) + 2 = 0$

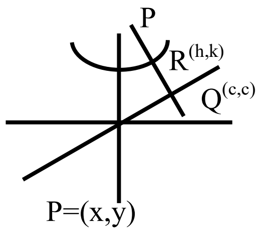
(2) $2(3x - y)^2 + (x - 3y) + 2 = 0$

(3) $(3x - y)^2 + 2(x - 3y) + 2 = 0$

(4) $2(x - 3y)^2 + (3x - y) + 2 = 0$

Official Ans. by NTA (2)

Sol.



$\frac{K-C}{h-C}=-1$

$C=\frac{h+K}{2}$ P(x,y)

$R=\left(\frac{x+C}{2},\frac{y+C}{2}\right)$

$R=\left(\frac{x}{2}+\frac{h}{4}+\frac{K}{4},\frac{y}{2}+\frac{h}{4}+\frac{k}{4}\right)$

$h=\frac{x}{2}+\frac{h}{4}+\frac{K}{4}$

$K=\frac{y}{2}+\frac{h}{4}+\frac{K}{4}$

$\Rightarrow x=\frac{3h}{2}-\frac{K}{2},y=\frac{3K}{2}-\frac{h}{2}$

$Y = 4x^2 + 1$

$\left(\frac{3k-h}{2}\right)=4\left(\frac{3h-k}{2}\right)^2+1$

18. The value of $k \in \mathbf{R}$, for which the following system of linear equations

$3x - y + 4z = 3,$

$x + 2y - 3z = -2,$

$6x + 5y + kz = -3,$

has infinitely many solutions, is :

(1) 3 (2) – 5 (3) 5 (4) –3

Official Ans. by NTA (2)

Sol. $\begin{vmatrix}3 & -1 & 4\\1 & 2 & -3\\6 & 5 & K\end{vmatrix}=0$

$\Rightarrow 3(2K + 15) + K + 18 - 28 = 0$

$\Rightarrow 7K + 35 = 0 \Rightarrow K = -5$

19. If sum of the first 21 terms of the series $\log_{9^{1/2}}x+\log_{9^{1/3}}x+\log_{9^{1/4}}x+.....$, where x > 0 is 504, then x is equal to

(1) 243 (2) 9 (3) 7 (4) 81

Official Ans. by NTA (4)

Sol. $s = 2\log_9 x + 3\log_9 x + ... + 22\log_9 x$

$s = \log_9 x\,(2 + 3 + ... + 22)$

$s=\log_9 x\left\{\frac{21}{2}(2+22)\right\}$

Given $252\log_9 x=504$

$\Rightarrow \log_9 x = 2 \Rightarrow x = 81$

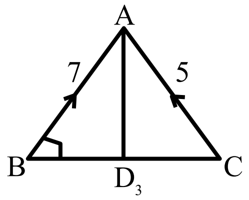
20. In a triangle ABC, if $|\overrightarrow{BC}|=3$, $|\overrightarrow{CA}|=5$ and $|\overrightarrow{BA}|=7$, then the projection of the vector $\overrightarrow{BA}$ on $\overrightarrow{BC}$ is equal to

(1) $\frac{19}{2}$ (2) $\frac{13}{2}$

(3) $\frac{11}{2}$ (4) $\frac{15}{2}$

Official Ans. by NTA (3)

Sol.



Projection of $\overrightarrow{BA}$

on $\overrightarrow{BC}$ is equal to

$= |\vec{BA}| \cos\angle ABC$

$= 7\left|\frac{7^2+3^2-5^2}{2\times7\times3}\right| = \frac{11}{2}$

SECTION-B

1. Let $A = \{a_{ij}\}$ be a 3 × 3 matrix, where

$$a_{ij} = \begin{cases} (-1)^{j-i} & \text{if } i < j, \\ 2 & \text{if } i = j, \\ (-1)^{i+j} & \text{if } i > j, \end{cases}$$

then det $(3\text{Adj}(2A^{-1}))$ is equal to ______.

Official Ans. by NTA (108)

Sol. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$|A| = 4$

$\left|3\text{adj}\left(2A^{-1}\right)\right| = \left|3.2^2\text{adj}\left(A^{-1}\right)\right|$

$= 12^3\left|\text{adj}\left(A^{-1}\right)\right| = 12^3\left|A^{-1}\right|^2 = \frac{12^3}{|A|^2} = \frac{12^3}{16} = 108$

2. The number of solutions of the equation

$\log_{(x+1)}(2x^2+7x+5) + \log_{(2x+5)}(x+1)^2 - 4 = 0$,

$x > 0$, is

Official Ans. by NTA (1)

Sol. $\log_{(x+1)}(2x^2+7x+5) + \log_{(2x+5)}(x+1)^2 - 4 = 0$

$\log_{(x+1)}(2x+5)(x+1) + 2\log_{(2x+5)}(x+1) = 4$

$\log_{(x+1)}(2x+5) + 1 + 2\log_{(2x+5)}(x+1) = 4$

Put $\log_{(x+1)}(2x+5) = t$

$t + \frac{2}{t} = 3 \Rightarrow t^2 - 3t + 2 = 0$

$t = 1, 2$

$\log_{(x+1)}(2x+5) = 1$ & $\log_{(x+1)}(2x+5) = 2$

$x + 1 = 2x + 3$ & $2x + 5 = (x+1)^2$

$x = -4$ (rejected) $x^2 = 4 \Rightarrow x = 2, -2$ (rejected)

So, $x = 2$

No. of solution = 1

3. Let a curve $y = y(x)$ be given by the solution of the differential equation

$\cos\left(\frac{1}{2}\cos^{-1}(e^{-x})\right)dx = \sqrt{e^{2x}-1}\,dy$

If it intersects y-axis at $y = -1$, and the intersection point of the curve with x-axis is $(\alpha, 0)$, then e^{α} is equal to ________.

Official Ans. by NTA (2)

Sol. $\cos\left(\frac{1}{2}\cos^{-1}(e^{-x})\right)dx = \sqrt{e^{2x}-1}\,dy$

Put $\cos^{-1}(e^{-x})\ \theta$, $\theta \in [0, \pi]$

$\cos\theta = e^{-x} \Rightarrow 2\cos^2\frac{\theta}{2} - 1 = e^{-x}$

$\cos\frac{\theta}{2} = \sqrt{\frac{e^{-x}+1}{2}} = \sqrt{\frac{e^x+1}{2e^x}}$

$\sqrt{\frac{e^x+1}{2e^x}}dx = \sqrt{e^{2x}-1}\,dy$

$\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{e^x}\sqrt{e^x-1}} = \int dy$

Put $e^x = t$, $\frac{dt}{dx} = e^x$

$\frac{1}{\sqrt{2}}\int\frac{dt}{e^x\sqrt{e^x}\sqrt{e^x-1}} = \int dy$

$\int\frac{dt}{t\sqrt{t^2-t}} = \sqrt{2}y$

Put $t=\frac{1}{z},\frac{dt}{dz}=-\frac{1}{z^2}$

$$\int\frac{-\frac{dz}{z^2}}{\frac{1}{z}\sqrt{\frac{1}{z^2}-\frac{1}{z}}}=\sqrt{2}y$$

$$-\int\frac{dz}{\sqrt{1-z}}=\sqrt{2}y$$

$$\frac{-2(1-z)^{1/2}}{-1}=\sqrt{2}y+c$$

$$2\left(1-\frac{1}{t}\right)^{1/2}=\sqrt{2}y+c$$

$$2\left(1-e^{-x}\right)^{1/2}=\sqrt{2y}+c\xrightarrow{(0,-1)}\Rightarrow c=\sqrt{2}$$

$2(1-e^{-x})^{1/2}=\sqrt{2}(y+1)$, passes through $(\alpha, 0)$

$$2(1-e^{-\alpha})^{1/2}=\sqrt{2}$$

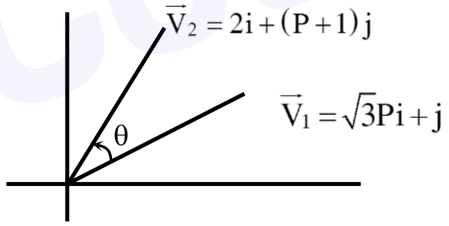
$$\sqrt{1-e^{-\alpha}}=\frac{1}{\sqrt{2}}\Rightarrow 1-e^{-\alpha}=\frac{1}{2}$$

$$e^{-\alpha}=\frac{1}{2}\Rightarrow e^{\alpha}=2$$

4. For $p>0$, a vector $\vec{v}_2=2\hat{i}+(p+1)\hat{j}$ is obtained by rotating the vector $\vec{v}_1=\sqrt{3}p\hat{i}+\hat{j}$ by an angle θ about origin in counter clockwise direction. If $\tan\theta=\frac{(\alpha\sqrt{3}-2)}{(4\sqrt{3}+3)}$, then the value of α is equal to ________.

Official Ans. by NTA (6)

Sol.



$$|\vec{V}_1|=|\vec{V}_2|$$

$3P^2+1=4+(P+1)^2$

$2P^2-2P-4=0\Rightarrow P^2-P-2=0$

$P=2,-1$ (rejected)

$$\cos\theta=\frac{\vec{V}_1\cdot\vec{V}_2}{|\vec{V}_1||\vec{V}_2|}=\frac{2\sqrt{3}P+(P+1)}{\sqrt{(P+1)^2+4}\sqrt{3P^2+1}}$$

$$\cos\theta=\frac{4\sqrt{3}+3}{\sqrt{13}\sqrt{13}}=\frac{4\sqrt{3}+3}{13}$$

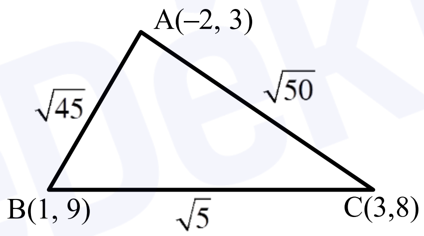
$$\tan\theta=\frac{\sqrt{112-24\sqrt{3}}}{4\sqrt{3}+3}=\frac{6\sqrt{3}-2}{4\sqrt{3}+3}=\frac{\alpha\sqrt{3}-2}{4\sqrt{3}+3}$$

$\Rightarrow\alpha=6$

5. Consider a triangle having vertices A(–2, 3), B(1, 9) and C(3, 8). If a line L passing through the circum-centre of triangle ABC, bisects line BC, and intersects y-axis at point $\left(0,\frac{\alpha}{2}\right)$, then the value of real number α is ________.

Official Ans. by NTA (9)

Sol.



$$(\sqrt{50})^2=(\sqrt{45})^2+(\sqrt{5})^2$$

$\angle B=90°$

Circum-center $=\left(\frac{1}{2},\frac{11}{2}\right)$

Mid point of BC $=\left(2,\frac{17}{2}\right)$

Line : $\left(y-\frac{11}{2}\right)=2\left(x-\frac{1}{2}\right)\Rightarrow y=2x+\frac{9}{2}$

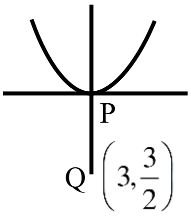
Passing though $\left(0,\frac{\alpha}{2}\right)$

$$\frac{\alpha}{2}=\frac{9}{2}\Rightarrow\alpha=9$$

6. If the point on the curve $y^2 = 6x$, nearest to the point $\left(3, \frac{3}{2}\right)$ is (α, β), then $2(\alpha + \beta)$ is equal to _______.

Official Ans. by NTA (9)

Sol.



$$P \equiv \left(\frac{3}{2}t^2, 3t\right)$$

Normal at point P

$$tx + y = 3t + \frac{3}{2}t^3$$

Passes through $\left(3, \frac{3}{2}\right)$

$$\Rightarrow 3t + \frac{3}{2} = 3t + \frac{3}{2}t^3$$

$$P \equiv \left(\frac{3}{2}, 3\right) = (\alpha, \beta)$$

$$\Rightarrow t^3 = 1 \Rightarrow t = 1$$

$$2(\alpha + \beta) = 2\left(\frac{3}{2} + 3\right) = 9$$

7. Let a function $g : [0, 4] \rightarrow \mathbf{R}$ be defined as

$$g(x) = \begin{cases} \max\limits_{0 \le t \le x} \{t^3 - 6t^2 + 9t - 3\}, & 0 \le x \le 3 \\ 4 - x & , 3 < x \le 4 \end{cases},$$

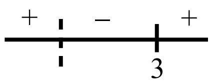
then the number of points in the interval (0, 4) where g(x) is NOT differentiable, is _______.

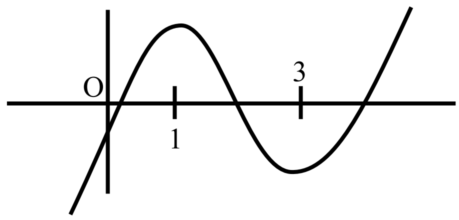
Official Ans. by NTA (1)

Sol. $f(x) = x^3 - 6x^2 + 9x - 3$

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$

$f(1) = 1 \; f(3) = -3$





$$g(x) = \begin{cases} f(x) & 0 \le x \le 1 \\ 0 & 1 \le x \le 3 \\ -1 & 3 < x \le 4 \end{cases}$$

g(x) is continuous

$$g'(x) = \begin{cases} 3(x-1)(x-3) & 0 \le x \le 1 \\ 0 & 1 \le x \le 3 \\ -1 & 3 < x \le 4 \end{cases}$$

g(x) is non-differentiable at x = 3

8. For $k \in N$, let

$$\frac{1}{\alpha(\alpha+1)(\alpha+2).......(\alpha+20)} = \sum_{K=0}^{20} \frac{A_k}{\alpha + k},$$

where $\alpha > 0$. Then the value of $100\left(\frac{A_{14}}{A_{13}}\right.$ is equal to _________.

Official Ans. by NTA (9)

Sol. $$\frac{1}{\alpha(\alpha+1).....(\alpha+20)} = \sum_{k=0}^{20} \frac{A_k}{\alpha + k}$$

$$A_{14} = \frac{1}{(-14)(-13).....(-1)(1).....(6)} = \frac{1}{14!\cdot 6!}$$

$$A_{15} = \frac{1}{(-15)(-14).....(-1)(1).....(5)} = \frac{1}{15!\cdot 5!}$$

$$A_{13} = \frac{1}{(-13).....(-1)(1).....(7)} = \frac{-1}{13!\cdot 7!}$$

$$\frac{A_{14}}{A_{13}} = \frac{1}{14!\cdot 6!} \times -13! \times 7! = \frac{-7}{14} = -\frac{1}{2}$$

$$\frac{A_{15}}{A_{13}} = -\frac{1}{15! \times 5!} \times -13! \times 7! = \frac{42}{15 \times 14} = \frac{1}{5}$$

$$100\left(\frac{A_{14}}{A_{13}} + \frac{A_{15}}{A_{13}}\right)^2 = 100\left(-\frac{1}{2} + \frac{1}{5}\right)^2 = 9$$

9. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 = 1$, $a_2 = 1$ and $a_{n+2} = 2a_{n+1} + a_n$ for all $n \geq 1$. Then the value of $47\sum_{n=1}^{\infty}\frac{a_n}{2^{3n}}$ is equal to ________.

Official Ans. by NTA (7)

Sol. $a_{n+2} = 2a_{n+1} + a_n$, let $\sum_{n=1}^{\infty}\frac{a_n}{8^n} = P$

Divide by 8^n we get

$$\frac{a_{n+2}}{8^n} = \frac{2a_{n+1}}{8^n} + \frac{a_n}{8^n}$$

$$\Rightarrow 64\frac{a_{n+2}}{8^{n+2}} = \frac{16a_{n+1}}{8^{n+1}} + \frac{a_n}{8^n}$$

$$64\sum_{n=1}^{\infty}\frac{a_{n+2}}{8^{n+2}} = 16\sum_{n=1}^{\infty}\frac{a_{n+1}}{8^{n+1}} + \sum_{n=1}^{\infty}\frac{a_n}{8^n}$$

$$64\left(P - \frac{a_1}{8} - \frac{a_2}{8^2}\right) = 16\left(P - \frac{a_1}{8}\right) + P$$

$$\Rightarrow 64\left(P - \frac{1}{8} - \frac{1}{64}\right) = 16\left(P - \frac{1}{8}\right) + P$$

$64P - 8 - 1 = 16P - 2 + P$

$47P = 7$

10. If $\lim_{x\to 0}\frac{\alpha x e^x - \beta \log_e(1+x) + \gamma x^2 e^{-x}}{x\sin^2 x} = 10$, $\alpha, \beta, \gamma \in$ **R**, then the value of $\alpha + \beta + \gamma$ is ________.

Official Ans. by NTA (3)

Sol. $$\lim_{x\to 0}\frac{\alpha x\left(1 + x + \frac{x^2}{2}\right) - \beta\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) + \gamma x^2(1-x)}{x^3}$$

$$\lim_{x\to 0}\frac{x(\alpha - \beta) + x^2\left(\alpha + \frac{\beta}{2} + \gamma\right) + x^3\left(\frac{\alpha}{2} - \frac{\beta}{3} - \gamma\right)}{x^3} = 10$$

For limit to exist

$$\alpha - \beta = 0,\ \alpha + \frac{\beta}{2} + \gamma = 0$$

$$\frac{\alpha}{2} - \frac{\beta}{3} - \gamma = 10 \quad \ldots\ldots(i)$$

$$\beta = \alpha,\ \gamma = -3\frac{\alpha}{2}$$

Put in (i)

$$\frac{\alpha}{2} - \frac{\alpha}{3} + \frac{3\alpha}{2} = 10$$

$$\frac{\alpha}{6} + \frac{3\alpha}{2} = 10 \Rightarrow \frac{\alpha + 9\alpha}{6} = 10$$

$\Rightarrow \alpha = 6$

$\alpha = 6, \beta = 6, \gamma = -9$

$\alpha + \beta + \gamma = 3$