MATHEMATICS

SECTION-A

1. The Boolean expression $(p \wedge \sim q) \Rightarrow (q \vee \sim p)$ is equivalent to :

(1) $q \Rightarrow p$ (2) $p \Rightarrow q$

(3) $\sim q \Rightarrow p$ (4) $p \Rightarrow \sim q$

Official Ans. by NTA (2)

Sol.

p	q	~p	~q	$p\wedge\sim q$	$q\vee\sim p$	$(p\wedge\sim q) \Rightarrow (q\vee\sim p)$	$p\Rightarrow q$
T	F	F	T	T	F	F	F
F	T	T	F	F	T	T	T
T	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T

$\therefore\ (p\wedge\sim q) \Rightarrow (q\vee\sim p)$

$\equiv p \Rightarrow q$

So, option (2) is correct.

2. Let a be a positive real number such that $\int_0^a e^{x-[x]}\,dx = 10e - 9$ where [x] is the greatest integer less than or equal to x. Then a is equal to :

(1) $10 - \log_e(1 + e)$ (2) $10 + \log_e 2$

(3) $10 + \log_e 3$ (4) $10 + \log_e(1+ e)$

Official Ans. by NTA (2)

Sol. $a > 0$

Let $n \le a < n + 1,\ n \in W$

$\therefore\ a = [a] + \{a\}$

$\Downarrow$ G.I.F $\quad \Downarrow$ Fractional part

Here $[a] = n$

Now, $\int_0^a e^{x-[x]}\,dx = 10e - 9$

$\Rightarrow \int_0^n e^{\{x\}}\,dx + \int_n^a e^{x-[x]}\,dx = 10e - 9$

$\therefore\ n\int_0^1 e^x\,dx + \int_n^a e^{x-n}\,dx = 10e - 9$

$\Rightarrow n(e-1) + (e^{a-n} - 1) = 10e - 9$

$\therefore\ \boxed{n = 0}$ and $\{a\} = \log_e 2$

So, $a = [a] + \{a\} = (10 + \log_e 2)$

$\Rightarrow$ Option (2) is correct.

3. The mean of 6 distinct observations is 6.5 and their variance is 10.25. If 4 out of 6 observations are 2, 4, 5 and 7, then the remaining two observations are:

(1) 10, 11 (2) 3, 18

(3) 8, 13 (4) 1, 20

Official Ans. by NTA (1)

Sol. Let other two numbers be a, (21 – a)

Now,

$$10.25 = \frac{\left(4 + 16 + 25 + 49 + a^2 + (21-a)^2\right)}{6} - (6.5)^2$$

(Using formula for variance)

$\Rightarrow 6(10.25) + 6(6.5)^2 = 94 + a^2 + (21 - a)^2$

$\Rightarrow a^2 + (21 - a)^2 = 221$

$\therefore\ a = 10$ and $(21 - a) = 21 - 10 = 11$

So, remaining two observations are 10, 11.

$\Rightarrow$ Option (1) is correct.

4. The value of the integral $\int_{-1}^{1} \log_e(\sqrt{1-x} + \sqrt{1+x})\,dx$ is equal to :

(1) $\frac{1}{2}\log_e 2 + \frac{\pi}{4} - \frac{3}{2}$ (2) $2\log_e 2 + \frac{\pi}{4} - 1$

(3) $\log_e 2 + \frac{\pi}{2} - 1$ (4) $2\log_e 2 + \frac{\pi}{2} - \frac{1}{2}$

Official Ans. by NTA (3)

Sol. Let $I = 2\int_0^1 \underbrace{\ln\left(\sqrt{1-x}+\sqrt{1+x}\right)}_{(I)} \underbrace{1}_{(II)}\, dx$

(I.B.P.)

$$\therefore\ I = 2\left[\left(x.\ln\left(\sqrt{1-x}+\sqrt{1-x}\right)\right)_0^1\right.$$

$$\left.-\int_0^1 x.\left(\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\right)\cdot\left(\frac{1}{2\sqrt{1+x}}-\frac{1}{2\sqrt{1-x}}\right)dx\right]$$

$$= 2\left(\ln\sqrt{2}-0\right)-\frac{2}{2}\int_0^1 \frac{x\sqrt{1-x}-\sqrt{1+x}\ dx}{\left(\sqrt{1-x}+\sqrt{1+x}\right)\sqrt{1-x^2}}$$

$$= (\log_e 2) - \int_0^1 \frac{x\cdot\left(2-2\sqrt{1-x^2}\right)}{-2x\sqrt{1-x^2}}dx$$

(After rationalisation)

$$= (\log_e 2) + \int_0^1\left(\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)dx$$

$$= (\log_e 2) + \left(\sin^{-1}x\right)_0^1 - 1$$

$$= \log_e 2 + \left(\frac{\pi}{2}-0\right) - 1$$

$$\therefore\ I = (\log_e 2) + \frac{\pi}{2} - 1$$

$\Rightarrow$ Option (3) is correct.

5. If α and β are the distinct roots of the equation $x^2 + (3)^{1/4}x + 3^{1/2} = 0$, then the value of $\alpha^{96}(\alpha^{12}-1) + \beta^{96}(\beta^{12}-1)$ is equal to :

(1) 56×3^{25} (2) 56×3^{24}

(3) 52×3^{24} (4) 28×3^{25}

Official Ans. by NTA (3)

Sol. As, $\left(\alpha^2+\sqrt{3}\right) = -(3)^{1/4}\cdot\alpha$

$\Rightarrow \left(\alpha^4+2\sqrt{3}\alpha^2+3\right) = \sqrt{3}\alpha^2$ (On squaring)

$\therefore\ (\alpha^4+3) = (-)\sqrt{3}\alpha^2$

$\Rightarrow \alpha^8+6\alpha^4+9 = 3\alpha^4$ (Again squaring)

$\therefore\ \alpha^8+3\alpha^4+9 = 0$

$\Rightarrow \boxed{\alpha^8 = -9-3\alpha^4}$

(Multiply by α^4)

So, $\alpha^{12} = -9\alpha^4 - 3\alpha^8$

$\therefore\ \alpha^{12} = -9\alpha^4 - 3(-9-3\alpha^4)$

$\Rightarrow \alpha^{12} = \cancel{-9\alpha^4} + 27 + \cancel{9\alpha^4}$

Hence, $\boxed{\alpha^{12} = (27)^2}$

$\Rightarrow (\alpha^{12})^8 = (27)^8$

$\Rightarrow \alpha^{96} = (3)^{24}$

Similarly $\beta^{96} = (3)^{24}$

$\therefore\ \alpha^{96}(\alpha^{12}-1)+\beta^{96}(\beta^{12}-1) = (3)^{24}\times 52$

$\Rightarrow$ Option (3) is correct.

6. Let $A = \begin{bmatrix} 2 & 3 \\ a & 0 \end{bmatrix}$, $a \in \mathbf{R}$ be written as P + Q where P is a symmetric matrix and Q is skew symmetric matrix. If det(Q) = 9, then the modulus of the sum of all possible values of determinant of P is equal to :

(1) 36 (2) 24 (3) 45 (4) 18

Official Ans. by NTA (1)

Sol. $A = \begin{bmatrix} 2 & 3 \\ a & 0 \end{bmatrix}, a \in R$

and $P = \dfrac{A+A^T}{2} = \begin{bmatrix} 2 & \frac{3+a}{2} \\ \frac{a+3}{2} & 0 \end{bmatrix}$

and $Q = \dfrac{A-A^T}{2} = \begin{bmatrix} 0 & \frac{3-a}{2} \\ \frac{a-3}{2} & 0 \end{bmatrix}$

As, det (Q) = 9

$\Rightarrow (a-3)^2 = 36$

$\Rightarrow a = 3 \pm 6$

$\therefore\ \boxed{a = 9, -3}$

$\therefore\ \det.(P) = \begin{vmatrix} 2 & \frac{3+a}{2} \\ \frac{a+3}{2} & 0 \end{vmatrix}$

$= 0 - \dfrac{(a-3)^2}{4} = 0$, for $a = -3$

$= 0 - \dfrac{(a-3)^2}{4} = -\dfrac{1}{4}(12)(12)$, for $a = 9$

$\therefore$ Modulus of the sum of all possible values of det. (P) = $|-36| + |0| = 36$ Ans.

$\Rightarrow$ Option (1) is correct

7. If z and ω are two complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \frac{3\pi}{2}$, then $\arg\left(\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right)$ is :

(Here arg(z) denotes the principal argument of complex number z)

(1) $\frac{\pi}{4}$ (2) $-\frac{3\pi}{4}$ (3) $-\frac{\pi}{4}$ (4) $\frac{3\pi}{4}$

Official Ans. by NTA (2)

Sol. As $|z\omega| = 1$

$\Rightarrow$ If $|z| = r$, then $|\omega| = \frac{1}{r}$

Let $\arg(z) = \theta$

$\therefore \arg(\omega) = \left(\theta - \frac{3\pi}{2}\right)$

So, $z = re^{i\theta}$

$\Rightarrow \bar{z} = re^{i(-\theta)}$

$\omega = \frac{1}{r}e^{i\left(\theta - \frac{3\pi}{2}\right)}$

Now, consider

$$\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega} = \frac{1-2e^{i\left(-\frac{3\pi}{2}\right)}}{1+3e^{i\left(-\frac{3\pi}{2}\right)}} = \left(\frac{1-2i}{1+3i}\right)$$

$$= \frac{(1-2i)(1-3i)}{(1+3i)(1-3i)} = -\frac{1}{2}(1+i)$$

$\therefore$ prin arg $\left(\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right)$

$= $ prin arg $\left(\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right)$

$= \left(-\frac{1}{2}(1+i)\right)$

$= -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}$

So, option (2) is correct.

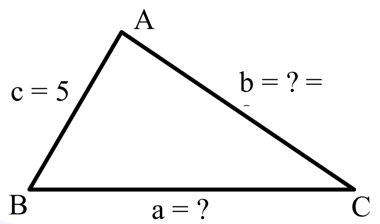
8. If in a triangle ABC, AB = 5 units, $\angle B = \cos^{-1}\left(\frac{3}{5}\right)$ and radius of circumcircle of ΔABC is 5 units, then the area (in sq. units) of ΔABC is :

(1) $10+6\sqrt{2}$ (2) $8+2\sqrt{2}$

(3) $6+8\sqrt{3}$ (4) $4+2\sqrt{3}$

Official Ans. by NTA (3)

Sol.



As, $\cos B = \frac{3}{5} \Rightarrow \boxed{B = 53°}$

As, $R = 5 \Rightarrow \frac{c}{\sin c} = 2R$

$\Rightarrow \frac{5}{10} = \sin c \Rightarrow \boxed{C = 30°}$

Now, $\frac{b}{\sin B} = 2R \Rightarrow \boxed{b = 2(5)\left(\frac{4}{5}\right) = 8}$

Now, by cosine formula

$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \frac{3}{5} = \frac{a^2 + 25 - 64}{2(5)a}$

$\Rightarrow a^2 - 6a - 3g = 0$

$\therefore a = \frac{6 \pm \sqrt{192}}{2} = \frac{6 \pm 8\sqrt{3}}{2}$

$\Rightarrow \boxed{3+4\sqrt{3}}$ (Reject $a = 3 - 4\sqrt{3}$)

Now, $\Delta = \frac{abc}{4R} = \frac{(3+4\sqrt{3})(8)(5)}{4(5)} = 2(3+4\sqrt{3})$

$\Rightarrow \Delta = (6 + 8\sqrt{3})$

$\Rightarrow$ Option (3) is correct.

9. Let [x] denote the greatest integer $\le$ x, where x $\in$ **R**. If the domain of the real valued function

$$f(x)=\sqrt{\frac{|[x]|-2}{|[x]|-3}}$$

is $(-\infty,a)\cup[b,c)\cup[4,\infty), a<b<c,$ then the value of $a+b+c$ is:

(1) 8 (2) 1

(3) –2 (4) –3

Official Ans. by NTA (3)

Sol. For domain,

$$\frac{|[x]|-2}{|[x]|-3}\ge 0$$

Case I : When $|[x]|-2\ge 0$

and $|[x]|-3>0$

$\therefore$ x $\in$ (–∞, –3) $\cup$ [4, ∞) .....(1)

Case II : When $|[x]|-2\le 0$

and $|[x]|-3<0$

$\therefore$ x $\in$ [–2, 3) .....(2)

So, from (1) and (2)

we get

Domain of function

= (–∞, –3) $\cup$ [–2, 3) $\cup$ [4, ∞)

$\therefore$ (a + b + c) = – 3 + (–2) + 3 = –2 (a < b < c)

$\Rightarrow$ Option (3) is correct.

10. Let y = y(x) be the solution of the differential equation $x\tan\left(\frac{y}{x}\right)dy=\left(y\tan\left(\frac{y}{x}\right)-x\right)dx,$

$-1\le x\le 1, y\left(\frac{1}{2}\right)=\frac{\pi}{6}.$ Then the area of the region bounded by the curves x = 0, x = $\frac{1}{\sqrt{2}}$ and y = y (x) in the upper half plane is:

(1) $\frac{1}{8}(\pi-1)$ (2) $\frac{1}{12}(\pi-3)$

(3) $\frac{1}{4}(\pi-2)$ (4) $\frac{1}{6}(\pi-1)$

Official Ans. by NTA (1)

Sol. We have

$$\frac{dy}{dx}=\frac{x\left(\frac{y}{x}\cdot\tan\frac{y}{x}-1\right)}{x\tan\frac{y}{x}}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}-\cot\left(\frac{y}{x}\right)$$

Put $\frac{y}{x}=v$

$\Rightarrow y=vn$

$$\therefore\frac{dy}{dx}=v+\frac{ndv}{dx}$$

Now, we get

$$v+n\frac{dv}{dx}=v-\cot(v)$$

$$\Rightarrow\int(\tan)dv=-\int\frac{dx}{x}$$

$$\therefore \ell n\left|\sec\left(\frac{y}{x}\right)\right|=-\ell n|x|+c$$

$$\text{As}\left(\frac{1}{2}\right)=\left(\frac{y}{x}\right)\Rightarrow\boxed{C=0}$$

$$\therefore \sec\left(\frac{y}{x}\right)=\frac{1}{x}$$

$$\Rightarrow\cos\left(\frac{y}{x}\right)=x$$

$$\therefore\boxed{y=x\cos^{-1}(x)}$$

So, required bounded area

$$=\int_0^{1/\sqrt{2}}\underset{(II)}{x}\,(\underset{(I)}{\cos^{-1}x})dx=\left(\frac{\pi-1}{8}\right)$$

(I.B.P.)

$\therefore$ option (1) is correct.

11. The coefficient of x^{256} in the expansion of $(1-x)^{101}(x^2+x+1)^{100}$ is:

(1) ${}^{100}C_{16}$ (2) ${}^{100}C_{15}$

(3) $-{}^{100}C_{16}$ (4) $-{}^{100}C_{15}$

Official Ans. by NTA (2)

Sol. $(1-x)^{100} \cdot (x^2+x+1)^{100} \cdot (1-x)$

$$=\left((1-x)(x^2+x+1)\right)^{100}(1-x)$$

$$=(1^3-x^3)^{100}(1-x)$$

$$=(1-x^3)^{100}(1-x)$$

$$=\underbrace{(1-x^3)^{100}}_{\text{No term of } x^{256}}-\underbrace{x(1-x^3)^{100}}_{\text{We find coefficient of } x^{255}}$$

Required coefficient $(-1)\times(-{}^{100}C_{85})$

$= {}^{100}C_{85} = {}^{100}C_{15}$

12. Let $A = [a_{ij}]$ be a 3×3 matrix, where

$$a_{ij}=\begin{cases}1 & , \text{ if } i=j\\ -x & , \text{ if } |i-j|=1\\ 2x+1 & , \text{ otherwise.}\end{cases}$$

Let a function $f: \mathbf{R} \square \mathbf{R}$ be defined as $f(x) = \det(A)$. Then the sum of maximum and minimum values of f on **R** is equal to:

(1) $-\frac{20}{27}$ (2) $\frac{88}{27}$

(3) $\frac{20}{27}$ (4) $-\frac{88}{27}$

Official Ans. by NTA (4)

Sol. $A=\begin{bmatrix}1 & -x & 2x+1\\ -x & 1 & -x\\ 2x+1 & -x & 1\end{bmatrix}$

$|A| = 4x^3 - 4x^2 - 4x = f(x)$

$f'(x) = 4(3x^2 - 2x - 1) = 0$

$\Rightarrow x = 1 \,;\, x = \frac{-1}{3}$

$\therefore \underbrace{f(1)=-4}_{\text{min}} \;; \underbrace{f\left(-\frac{1}{3}\right)=\frac{20}{27}}_{\text{max.}}$

$$\text{Sum} = -4+\frac{20}{27}=-\frac{88}{27}$$

13. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}\cdot\vec{c}=|\vec{c}|, |\vec{c}-\vec{a}|=2\sqrt{2}$ and the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$ is $\frac{\pi}{6}$, then the value of $\left|(\vec{a}\times\vec{b})\times\vec{c}\right|$

(1) $\frac{2}{3}$ (2) 4 (3) 3 (4) $\frac{3}{2}$

Official Ans. by NTA (4)

Sol. $|\vec{a}| = 3 = a \,;\, \vec{a}.\vec{c} = c$

Now $|\vec{c}-\vec{a}| = 2\sqrt{2}$

$\Rightarrow c^2 + a^2 - 2\vec{c}.\vec{a} = 8$

$\Rightarrow c^2 + 9 - 2(c) = 8$

$\Rightarrow c^2 - 2c + 1 = 0 \Rightarrow c = 1 = |\vec{c}|$

Also, $\vec{a}\times\vec{b} = 2\hat{i}-2\hat{j}+\hat{k}$

Given $(\vec{a}\times\vec{b}) = |\vec{a}\times\vec{b}||\vec{c}|\sin\frac{\pi}{6}$

$= (3)(1)(1/2)$

$= 3/2$

14. The number of real roots of the equation

$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{4} \text{ is :}$$

(1) 1 (2) 2

(3) 4 (4) 0

Official Ans. by NTA (4)

Sol. $\tan^{-1}\sqrt{x^2+x}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{4}$

For equation to be defined,

$x^2 + x \geq 0$

$\Rightarrow x^2 + x + 1 \geq 1$

$\therefore$ only possibility that the equation is defined

$x^2 + x = 0 \quad \Rightarrow x = 0; x = -1$

None of these values satisfy

$\therefore$ No of roots = 0

15. Let $y = y(x)$ be the solution of the differential equation $e^x\sqrt{1-y^2}\,dx+\left(\frac{y}{x}\right)dy=0, y(1)=-1.$

Then the value of $(y(3))^2$ is equal to:

(1) $1-4e^3$ (2) $1-4e^6$

(3) $1+4e^3$ (4) $1+4e^6$

Official Ans. by NTA (2)

Sol. $e^x\sqrt{1-y^2}dx + \frac{y}{x}dy = 0$

$\Rightarrow e^x\sqrt{1-y^2}dx + \frac{-y}{x}dy$

$\Rightarrow \int \frac{-y}{\sqrt{1-y^2}}\,dy = \int \underset{II}{e^x}\ \ \underset{1}{x}\ dx$

$\Rightarrow \sqrt{1-y^2} = e^x(x-1)+c$

Given : At x = 1, y = – 1

$\Rightarrow 0 = 0 + c \Rightarrow c = 0$

$\therefore \sqrt{1-y^2} = e^x(x-1)$

At x = 3 $\quad 1 - y^2 = (e^3\ 2)^2 \Rightarrow y^2 = 1 - 4e^6$

16. Let 'a' be a real number such that the function $f(x) = ax^2 + 6x - 15$, $x \in \mathbf{R}$ is increasing in $\left(-\infty, \frac{3}{4}\right)$ and decreasing in $\left(\frac{3}{4}, \infty\right)$. Then the function $g(x) = ax^2 - 6x + 15$, $x \in \mathbf{R}$ has a:

(1) local maximum at $x = -\frac{3}{4}$

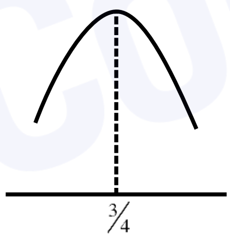
(2) local minimum at $x = -\frac{3}{4}$

(3) local maximum at $x = \frac{3}{4}$

(4) local minimum at $x = \frac{3}{4}$

Official Ans. by NTA (1)

Sol.



$\frac{-B}{2A} = \frac{3}{4}$

$\Rightarrow \frac{-(6)}{2a} = \frac{3}{4}$

$\Rightarrow a = \frac{-6\times 4}{6} \Rightarrow a = -4$

$\therefore\ g(x) = 4x^2 - 6x + 15$

Local max. at $x = \frac{-B}{2A} = -\frac{(-6)}{2(-4)}$

$= \frac{-3}{4}$

17. Let a function $f : \mathbf{R} \to \mathbf{R}$ be defined as

$$f(x) = \begin{cases} \sin x - e^x & \text{if } x \le 0 \\ a + [-x] & \text{if } 0 < x < 1 \\ 2x - b & \text{if } x \ge 1 \end{cases}$$

Where [x] is the greatest integer less than or equal to x. If f is continuous on **R**, then (a + b) is equal to:

(1) 4 (2) 3

(3) 2 (4) 5

Official Ans. by NTA (2)

Sol. Continuous at x = 0

$f(0^+) = f(0^-) \Rightarrow a - 1 = 0 - e^0$

$\Rightarrow a = 0$

Continuous at x = 1

$f(1^+) = f(1^-)$

$\Rightarrow 2(1) - b = a + (-1)$

$\Rightarrow b = 2 - a + 1 \Rightarrow b = 3$

$\therefore a + b = 3$

18. Words with or without meaning are to be formed using all the letters of the word EXAMINATION. The probability that the letter M appears at the fourth position in any such word is:

(1) $\frac{1}{66}$ (2) $\frac{1}{11}$ (3) $\frac{1}{9}$ (4) $\frac{2}{11}$

Official Ans. by NTA (2)

Sol. AAEIIMNNOTX

--------M----------

Total words with M at fourth Place = $\frac{10!}{2!2!2!}$

Total words = $\frac{11!}{2!2!2!}$

Required probability = $\frac{10!}{11!} = \frac{1}{11}$

19. The probability of selecting integers $a \in [-5, 30]$ such that $x^2 + 2(a + 4)x - 5a + 64 > 0$, for all $x \in \mathbf{R}$, is:

(1) $\frac{7}{36}$ (2) $\frac{2}{9}$ (3) $\frac{1}{6}$ (4) $\frac{1}{4}$

Official Ans. by NTA (2)

Sol. $D < 0$

$\Rightarrow 4(a + 4)^2 - 4(-5a + 64) < 0$

$\Rightarrow a^2 + 16 + 8a + 5a - 64 < 0$

$\Rightarrow a^2 + 13a - 48 < 0$

$\Rightarrow (a + 16)(a - 3) < 0$

$\Rightarrow a \in (-16, 3)$

$\therefore$ Possible a : $\{-5, -4, ..., 3\}$

$\therefore$ Required probability $= \frac{8}{36}$

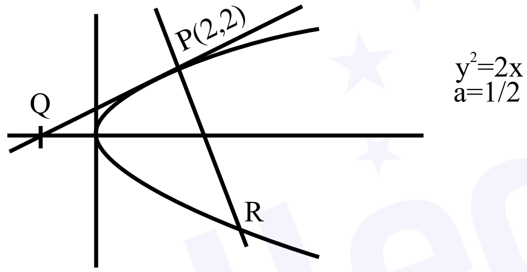
$= \frac{2}{9}$

20. Let the tangent to the parabola $S : y^2 = 2x$ at the point $P(2, 2)$ meet the x-axis at Q and normal at it meet the parabola S at the point R. Then the area (in sq. units) of the triangle PQR is equal to:

(1) $\frac{25}{2}$ (2) $\frac{35}{2}$ (3) $\frac{15}{2}$ (4) 25

Official Ans. by NTA (1)

Sol.



Tangent at P : $y(2) = 2(1/2)(x + 2)$

$\Rightarrow 2y = x + 2$

$\therefore\ Q = (-2, 0)$

Normal at P : $y - 2 = -\frac{(2)}{2.\frac{1}{2}}(x - 2)$

$\Rightarrow y - 2 = -2(x - 2)$

$\Rightarrow y = 6 - 2x$

$\therefore$ Solving with $y^2 = 2x \Rightarrow R\left(\frac{9}{2} - 3\right)$

$$\therefore \text{Ar}(\Delta PQR) = \frac{1}{2}\begin{vmatrix} 2 & 2 & 1 \\ -2 & 1 & 1 \\ \frac{9}{2} & 3- & 1 \end{vmatrix}$$

$= \frac{25}{2}$ sq.units

SECTION-B

1. Let $\vec{a}, \vec{b}, \vec{c}$ be three mutually perpendicular vectors of the same magnitude and equally inclined at an angle θ, with the vector $\vec{a} + \vec{b} + \vec{c}$. Then

$36\cos^2 2\theta$ is equal to ________.

Official Ans. by NTA (4)

Sol. $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{a}.\vec{c} + \vec{b}.\vec{c})$

$= 3$

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}$

$\vec{a}.(\vec{a} + \vec{b} + \vec{c}) = |\vec{a}| + |\vec{a} + \vec{b} + \vec{c}|\cos\theta$

$\Rightarrow 1 = \sqrt{3}\cos\theta$

$\Rightarrow \cos 2\theta = -\frac{1}{3}$

$\Rightarrow 36\cos^2 2\theta = \boxed{4}$

2. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = 7A^{20} - 20A^7 + 2I$, where I is an identity matrix of order 3×3. If $B = [b_{ij}]$, then b_{13} is equal to ________.

Official Ans. by NTA (910)

Sol. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = I + C$

where $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$C^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$

$C^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = C^4 = C^5 = ...$

$B = 7A^{20} - 20A^7 + 2I$

$= 7(I + C)^{20} - 20(I + C)^7 + 2I$

$= 7(I + 20C + {}^{20}C_2 C^2) - 20(I + 7C + {}^7C_2 C^2) + 2I$

So

$b_{13} = 7 \times {}^{20}C_2 - 20 \times {}^7C_2 = \boxed{910}$

3. Let P be a plane passing through the points (1, 0, 1), (1, –2, 1) and (0, 1, –2). Let a vector $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ be such that $\vec{a}$ is parallel to the plane P, perpendicular to $\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and $\vec{a}\cdot\left(\hat{i}+\hat{j}+2\hat{k}\right)=2$, then $(\alpha - \beta + \gamma)^2$ equals ________.

Official Ans. by NTA (81)

Sol. Equation of plane :

$$\begin{vmatrix} x-1 & y-0 & z-1 \\ 1-1 & 2 & 1-1 \\ 1-0 & 0-1 & 1+2 \end{vmatrix} = 0$$

$\Rightarrow 3x - z - 2 = 0$

$\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} \parallel$ to $3x - z - 2 = 0$

$\Rightarrow \boxed{3\alpha - 8 = 0}$ .....(1)

$\vec{a} \perp i + 2\hat{j} + 3\hat{k}$

$\Rightarrow \alpha + 2\beta + 38 = 0$ ..... (2)

$\vec{a}.(\hat{i}+\hat{j}+2\hat{k}) = 0$

$\Rightarrow \alpha + \beta + 28 = 2$ ..... (3)

on solving 1, 2 & 3

$\alpha = 1,\ \beta = -5,\ 8 = 3$

So $(\alpha - \beta + 8) = \boxed{81}$

4. The number of rational terms in the binomial expansion of $\left(4^{\frac{1}{4}} + 5^{\frac{1}{6}}\right)^{120}$ is __________.

Official Ans. by NTA (21)

Sol. $\left(4^{1/4} + 5^{1/6}\right)^{120}$

$T_{r+1} = {}^{120}C_r (2^{1/2})^{120-r}(5)^{r/6}$

for rational terms $r = 6\lambda \quad 0 \le r \le 120$

so total no of forms are 21.

5. If the shortest distance between the lines $\vec{r_1} = \alpha\hat{i} + 2\hat{j} + 2\hat{k} + \lambda\left(\hat{i} - 2\hat{j} + 2\hat{k}\right)$, $\lambda \in \mathbf{R}$, $\alpha > 0$ and $\vec{r_2} = -4\hat{i} - \hat{k} + \mu\left(3\hat{i} - 2\hat{j} - 2\hat{k}\right)$, $\mu \in \mathbf{R}$ is 9, then α is equal to _________.

Official Ans. by NTA (6)

Sol. If $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \lambda\vec{d}$

then shortest distance between two lines is

$$L = \frac{(\vec{a}-\vec{c}).(\vec{b}\times\vec{d})}{|b \times d|}$$

$\therefore\ \vec{a} - \vec{c} = ((\alpha+4)\hat{i} + 2\hat{j} + 3\hat{k})$

$$\frac{\vec{b}\times\vec{d}}{|b\times d|} = \frac{(2\hat{i}+2\hat{j}+\hat{k})}{3}$$

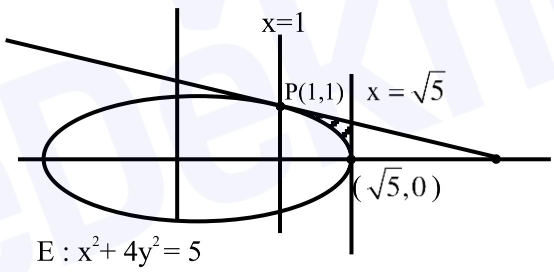
$$\therefore\ ((\alpha+4)\hat{i}+2\hat{j}+3\hat{k}).\frac{(2\hat{i}+2\hat{j}+\hat{k})}{3} = 9$$

or $\alpha = 6$

6. Let T be the tangent to the ellipse E : $x^2 + 4y^2 = 5$ at the point P(1, 1). If the area of the region bounded by the tangent T, ellipse E, lines x = 1 and $x = \sqrt{5}$ is $\alpha\sqrt{5} + \beta + \gamma\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$, then $|\alpha+\beta+\gamma|$ is equal to __________.

Official Ans. by NTA (1)

Sol.



Tangent at P : x + 4y = 5

Required Area

$$= \int_1^{\sqrt{5}} \left(\frac{5-x}{4} - \frac{\sqrt{5-x^2}}{2}\right) dx$$

$$= \left[\frac{5x}{4} - \frac{x^2}{8} - \frac{x}{4}\sqrt{5-x^2} - \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_1^{\sqrt{5}}$$

$$= \frac{5}{4}\sqrt{5} - \frac{5}{4} - \frac{5}{4}\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

It we assume $\alpha, \beta, \gamma, \in Q$ (Not given in question)

then $\alpha = \frac{5}{4}, \beta = -\frac{5}{4}$ & $\gamma = -\frac{5}{4}$

$|\alpha + \beta + \gamma| = 1.25$

7. Let a, b, c, d be in arithmetic progression with common difference λ. If

$$\begin{vmatrix} x+a-c & x+b & x+a \\ x-1 & x+c & x+b \\ x-b+d & x+d & x+c \end{vmatrix} = 2,$$

then value of λ^2 is equal to ____________.

Official Ans. by NTA (1)

Sol. $\begin{vmatrix} x+a-c & x+b & x+a \\ x-1 & x+c & x+b \\ x-b+d & x+d & x+c \end{vmatrix} = 2$

$C_2 \to C_2 - C_3$

$$\Rightarrow \begin{vmatrix} x-2\lambda & \lambda & x+a \\ x-1 & \lambda & x+b \\ x+2\lambda & \lambda & x+c \end{vmatrix} = 2$$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$\Rightarrow \lambda \begin{vmatrix} x-2\lambda & 1 & x+a \\ 2\lambda-1 & 0 & \lambda \\ 4\lambda & 0 & 2\lambda \end{vmatrix} = 2$$

$\Rightarrow 1\,(4\lambda^2 - 4\lambda^2 + 2\lambda) = 2$

$\Rightarrow \boxed{\lambda^2 = 1}$

8. There are 15 players in a cricket team, out of which 6 are bowlers, 7 are batsmen and 2 are wicketkeepers. The number of ways, a team of 11 players be selected from them so as to include at least 4 bowlers, 5 batsmen and 1 wicketkeeper, is__________.

Official Ans. by NTA (777)

Sol. 15 : Players

6 : Bowlers

7 : Batsman

2 : Wicket keepers

Total number of ways for :

at least 4 bowlers, 5 batsman & 1 wicket keeper

$= {}^6C_4({}^7C_6 \times {}^2C_1 + {}^7C_5 \times {}^2C_2) + {}^6C_5 \times {}^7C_5 \times {}^2C_1$

$= \boxed{777}$

9. Let y = mx + c, m > 0 be the focal chord of $y^2 = -64x$, which is tangent to $(x + 10)^2 + y^2 = 4$. Then, the value of $4\sqrt{2}\,(m + c)$ is equal to_______.

Official Ans. by NTA (34)

Sol. $y^2 = -64x$

focus : (–16, 0)

y = mx + c is focal chord

$\Rightarrow c = 16m$ ....(1)

y = mx + c is tangent to $(x + 10)^2 + y^2 = 4$

$\Rightarrow y = m(x+10) \pm 2\sqrt{1+m^2}$

$\Rightarrow c = 10m \pm 2\sqrt{1+m^2}$

$\Rightarrow 16m = 10m \pm 2\sqrt{1+m^2}$

$\Rightarrow 6m = 2\sqrt{1+m^2}$ $(m > 0)$

$\Rightarrow 9m^2 = 1 + m^2$

$\Rightarrow m = \dfrac{1}{2\sqrt{2}}$ & $c = \dfrac{8}{\sqrt{2}}$

$4\sqrt{2}(m+c) = 4\sqrt{2}\left(\dfrac{17}{2\sqrt{2}}\right) = \boxed{34}$

10. If the value of $\lim\limits_{x\to 0}\left(2 - \cos x\sqrt{\cos 2x}\right)^{\left(\frac{x+2}{x^2}\right)}$ is equal to e^a, then a is equal to ________.

Official Ans. by NTA (3)

Sol. $\lim\limits_{x\to 0}\left(2 - \cos x\sqrt{\cos x}\right)^{\frac{x+2}{x^2}}$

form: 1^∞

$= e^{\lim\limits_{x\to 0}\left(\frac{1-\cos x\sqrt{\cos 2x}}{x^2}\right)\times(x+2)}$

Now $\operatorname{limt}\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$

=

$$\operatorname{limt}\limits_{x\to 0}\frac{\sin x\sqrt{\cos 2x} - \cos x \times \dfrac{1}{2\sqrt{\cos 2x}} \times (-2\sin 2x)}{2x}$$

(by L' Hospital Rule)

$\operatorname{limt}\limits_{x\to 0}\dfrac{\sin x\cos 2x + \sin 2x.\cos x}{2x}$

$= \dfrac{1}{2} + 1 = \dfrac{3}{2}$

So, $e^{\operatorname{limt}\limits_{x\to 0}\left(\frac{1-\cos x\sqrt{\cos 2x}}{x^2}\right)(x+2)}$

$= e^{\frac{3}{2}\times 2} = e^3$

$\Rightarrow \boxed{a = 3}$