MATHEMATICS

SECTION-A

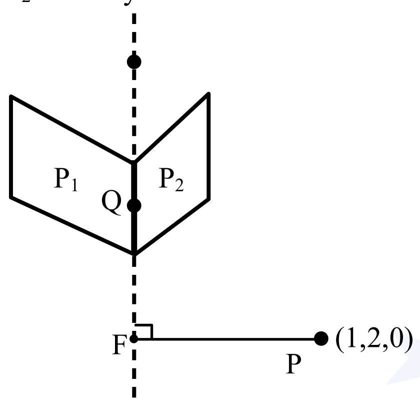
1. Let L be the line of intersection of planes $\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=2$ and $\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)=2$. If $P(\alpha,\beta,\gamma)$ is the foot of perpendicular on L from the point (1,2,0), then the value of $35(\alpha+\beta+\gamma)$ is equal to :

(1) 101 (2) 119 (3) 143 (4) 134

Official Ans. by NTA (2)

Sol. $P_1 : x - y + 2z = 2$

$P_2 = 2x + y - 3 = 2$



Let line of Intersection of planes P_1 and P_2 cuts xy plane in point Q.

$\Rightarrow$ z–coordinate of point Q is zero

$$\Rightarrow \left.\begin{matrix} x-y=2 \\ \text{and } 2x+y=2 \end{matrix}\right\} \Rightarrow x=\frac{4}{3}, y=\frac{-2}{3}$$

$$\Rightarrow Q\left(\frac{4}{3},\frac{-2}{3},0\right)$$

Vector parallel to the line of intersection

$$\vec{a}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}=-\hat{i}+5\hat{j}+3\hat{k}$$

Equation of Line of intersection

$$\frac{x-\frac{4}{3}}{-1}=\frac{y+\frac{2}{3}}{5}=\frac{z-0}{3}=\lambda(\text{say})$$

Let coordinates of foot of perpendicular be

$$F\left(-\lambda+\frac{4}{3},5\lambda-\frac{2}{3},3\lambda\right)$$

$$\overrightarrow{PF}=\left(-\lambda+\frac{1}{3}\right)\hat{i}+\left(5\lambda-\frac{8}{3}\right)\hat{j}+(3\lambda)\hat{k}$$

$$\overrightarrow{PF}\cdot\vec{a}=0$$

$$\Rightarrow \lambda-\frac{1}{3}+25\lambda\frac{-40}{3}+9\lambda=0$$

$$\Rightarrow 35\lambda=\frac{41}{3} \Rightarrow \boxed{\lambda=\frac{41}{105}}$$

Now, $\alpha=-\lambda+\frac{4}{3}, \beta=5\lambda-\frac{2}{3}, \gamma=3\lambda$

$$\Rightarrow \alpha+\beta+\gamma = 7\lambda+\frac{2}{3}$$

$$=7\left(\frac{41}{105}\right)+\frac{2}{3}$$

$$=\frac{51}{15}$$

$$\Rightarrow 35(\alpha+\beta+\gamma)=\frac{51}{15}\times 35=119$$

2. Let S_n denote the sum of first n-terms of an arithmetic progression. If $S_{10} = 530$, $S_5 = 140$, then $S_{20} - S_6$ is equal to :

(1) 1862 (2) 1842 (3) 1852 (4) 1872

Official Ans. by NTA (1)

Sol. $S_{10}=530 \Rightarrow \frac{10}{2}\{2a+9d\}=530$

$\Rightarrow 2a + 9d = 106$ ….(1)

and $S_5 = 140 \Rightarrow \frac{5}{2}\{2a+4d\}=140$

$\Rightarrow 2a + 4d = 56$ …..(2)

$\Rightarrow 5d = 50 \Rightarrow \boxed{d=10} \Rightarrow \boxed{a=8}$

Now, $S_{20}-S_6=\frac{20}{2}\{2a+19d\}-\frac{6}{2}\{2a+5d\}$

$= 14a + 175d$

$= (14 \times 8) + (175 \times 10)$

$= 1862$

3. Let $f:\mathbf{R}\to\mathbf{R}$ be defined as

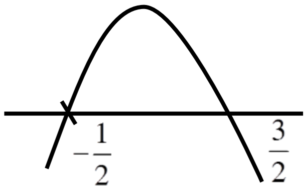
$$f(x)=\begin{cases} -\frac{4}{3}x^3+2x^2+3x & , \quad x>0 \\ 3xe^x & , \quad x\le 0 \end{cases}.$$ Then f is increasing function in the interval

(1) $\left(-\frac{1}{2},2\right)$ (2) (0,2)

(3) $\left(-1,\frac{3}{2}\right)$ (4) (–3, –1)

Official Ans. by NTA (3)

Sol. $f'(x)\begin{cases} -4x^2+4x+3 & x>0 \\ 3e^x(1+x) & x\le 0 \end{cases}$



For $x > 0$, $f'(x) = -4x^2 + 4x + 3$

$f(x)$ is increasing in $\left(-\frac{1}{2},\frac{3}{2}\right)$

For $x \le 0$, $f'(x) = 3e^x(1 + x)$

$f'(x) > 0\ \forall\ x \in (-1, 0)$

$\Rightarrow f(x)$ is increasing in $(-1, 0)$

So, in complete domain, $f(x)$ is increasing in

$\left(-1,\frac{3}{2}\right)$

4. Let $y = y(x)$ be the solution of the differential equation $\operatorname{cosec}^2x\,dy + 2dx = (1 + y\cos 2x)\operatorname{cosec}^2x\,dx$, with $y\left(\frac{\pi}{4}\right)=0$. Then, the value of $(y(0) + 1)^2$ is equal to :

(1) $e^{1/2}$ (2) $e^{-1/2}$ (3) e^{-1} (4) e

Official Ans. by NTA (3)

Sol. $\frac{dy}{dx}+2\sin^2 x=1+y\cos 2x$

$\Rightarrow \frac{dy}{dx}+(-\cos 2x)y=\cos 2x$

$\text{I.F.}=e^{\int -\cos 2x\,dx}=e^{-\frac{\sin 2x}{2}}$

Solution of D.E.

$y\left(e^{-\frac{\sin 2x}{2}}\right)=\int(\cos 2x)\left(e^{-\frac{\sin 2x}{2}}\right)dx+c$

$\Rightarrow y\left(e^{-\frac{\sin 2x}{2}}\right)=-e^{-\frac{\sin 2x}{2}}+c$

Given

$y\left(\frac{\pi}{4}\right)=0$

$\Rightarrow 0=-e^{-1/2}+c\Rightarrow \boxed{c=e^{-1/2}}$

$\Rightarrow y\left(e^{-\frac{\sin 2x}{2}}\right)=-e^{-\frac{\sin 2x}{2}}+e^{-1/2}$

at $x = 0$

$y=-1+e^{-1/2}$

$\Rightarrow y(0)=-1+e^{-1/2}\Rightarrow (y(0)+1)^2=e^{-1}$

5. Four dice are thrown simultaneously and the numbers shown on these dice are recorded in 2×2 matrices. The probability that such formed matrices have all different entries and are non-singular, is :

(1) $\frac{45}{162}$ (2) $\frac{23}{81}$ (3) $\frac{22}{81}$ (4) $\frac{43}{162}$

Official Ans. by NTA (4)

Sol. $A=\begin{vmatrix} a & b \\ c & d\end{vmatrix}$ $|A| = ad - bc$

Total case $= 6^4$

For non-singular matrix $|A| \ne 0 \Rightarrow ad - bc \ne 0$

$\Rightarrow ad \ne bc$

And a, b, c, d are all different numbers in the set $\{1, 2, 3, 4, 5, 6\}$

Now for $ad = bc$

(i) $6 \times 1 = 2 \times 3$

$\left.\begin{matrix}\Rightarrow a=6,\ b=2,\ c=3,\ d=1 \\ \text{or}\ a=1,\ b=2,\ c=3,\ d=6 \\ \vdots\end{matrix}\right\}$ 8 such cases

(ii) $6 \times 2 = 3 \times 4$

$\left.\begin{matrix}\Rightarrow a=6,\ b=3,\ c=4,\ d=2 \\ \text{or}\ a=2,\ b=3,\ c=4,\ d=6 \\ \vdots\end{matrix}\right\}$ 8 such cases

favourable cases

$= {}^6C_4\,\underline{|4} - 16$

required probability

$=\frac{{}^6C_4\,\underline{|4}-16}{6^4}=\frac{43}{162}$

6. Let a vector $\vec{a}$ be coplanar with vectors $\vec{b}=2\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}+\hat{k}$. If $\vec{a}$ is perpendicular to $\vec{d}=3\hat{i}+2\hat{j}+6\hat{k}$, and $|\vec{a}|=\sqrt{10}$. Then a possible value of $[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\ \vec{c}\ \vec{d}]$ is equal to :

(1) – 42 (2) – 40 (3) – 29 (4) – 38

Official Ans. by NTA (1)

Sol. $\vec{a}=\lambda\vec{b}+\mu\vec{c}=\hat{i}(2\lambda+\mu)+\hat{j}(\lambda-\mu)+\hat{k}(\lambda+\mu)$

$\vec{a}\cdot\vec{d}=0=3(2\lambda+\mu)+2(\lambda-\mu)+6(\lambda+\mu)$

$\Rightarrow 14\lambda+7\mu=0\Rightarrow \mu=-2\lambda$

$\Rightarrow \vec{a}=(0)\hat{i}-3\lambda\hat{j}+(-\lambda)\hat{k}$

$\Rightarrow |\vec{a}|=\sqrt{10}|\lambda|=\sqrt{10}\ \Rightarrow |\lambda|=1$

$\Rightarrow \lambda = 1$ or -1

$[\vec{a}\,\vec{b}\,\vec{c}]=0$

$[\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{d}]+[\vec{a}\,\vec{c}\,\vec{d}]=[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{d}]$

$$=\begin{vmatrix}0 & -3\lambda & \lambda\\ 3 & 0 & 2\\ 3 & 2 & 6\end{vmatrix}$$

$=3\lambda(12)+\lambda(6)=42\lambda=-42$

7. If $\int_0^{100\pi}\frac{\sin^2 x}{e^{\left(\frac{x}{\pi}-\left[\frac{x}{\pi}\right]\right)}}dx=\frac{\alpha\pi^3}{1+4\pi^2},\alpha\in\mathbf{R}$ where $[x]$ is the greatest integer less than or equal to x, then the value of α is :

(1) $200(1-e^{-1})$ (2) $100(1-e)$

(3) $50(e-1)$ (4) $150(e^{-1}-1)$

Official Ans. by NTA (1)

Sol. $I=\int_0^{100\pi}\frac{\sin^2 x}{e^{\{x/\pi\}}}dx=100\int_0^{\pi}\frac{\sin^2 x}{e^{x/\pi}}dx$

$100\int_0^{\pi}e^{-x/\pi}\frac{(1-\cos 2x)}{2}dx$

$=50\left\{\int_0^{\pi}e^{-x/\pi}dx-\int_0^{\pi}e^{-x/\pi}\cos 2x\,dx\right\}$

$I_1=\int_0^{\pi}e^{-x/\pi}dx=\left[-\pi e^{-x/\pi}\right]_0^{\pi}=\pi(1-e^{-1})$

$I_2=\int_0^{\pi}e^{-x/\pi}\cos 2x\,dx$

$=-\pi e^{-x/\pi}\cos 2x\Big]_0^{\pi}-\int -\pi e^{-x/\pi}(-2\sin 2x)dx$

$=\pi(1-e^{-1})-2\pi\int_0^{\pi}e^{-x/\pi}\sin 2x\,dx$

$=\pi(1-e^{-1})-2\pi\left\{-\pi e^{-x/\pi}\sin 2x\Big]_0^{\pi}-\int_0^{\pi}-\pi e^{-x/\pi}2\cos 2x dx\right\}$

$=\pi(1-e^{-1})-4\pi^2 I_2$

$\Rightarrow I_2=\frac{\pi(1-e^{-1})}{1+4\pi^2}$

$\therefore I=50\left\{\pi(1-e^{-1})-\frac{\pi(1-e^{-1})}{1+4\pi^2}\right\}$

$=\frac{200(1-e^{-1})\pi^3}{1+4\pi^2}$

8. Let three vectors $\vec{a},\vec{b}$ and $\vec{c}$ be such that $\vec{a}\times\vec{b}=\vec{c},\vec{b}\times\vec{c}=\vec{a}$ and $|\vec{a}|=2$. Then which one of the following is **not** true ?

(1) $\vec{a}\times((\vec{b}+\vec{c})\times(\vec{b}-\vec{c}))=\vec{0}$

(2) Projection of $\vec{a}$ on $(\vec{b}\times\vec{c})$ is 2

(3) $[\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{c}\ \ \vec{a}\ \ \vec{b}]=8$

(4) $|3\vec{a}+\vec{b}-2\vec{c}|^2=51$

Official Ans. by NTA (4)

Sol. (1) $\vec{a}\times((\vec{b}+\vec{c})\times(\vec{b}-\vec{c}))$

$=\vec{a}(-\vec{b}\times\vec{c}+\vec{c}\times\vec{b})=-2(\vec{a}\times(\vec{b}\times\vec{c}))$

$=-2(\vec{a}\times\vec{a})=\vec{0}$

(2) Projection of $\vec{a}$ on $\vec{b}\times\vec{c}$

$=\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}=\frac{\vec{a}\cdot\vec{a}}{|\vec{a}|}=|\vec{a}|=2$

(3) $[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{c}\ \vec{a}\ \vec{b}]=2[\vec{a}\ \vec{b}\ \vec{c}]=2\vec{a}\cdot(\vec{b}\times\vec{c})$

$=2\vec{a}.\vec{a}=2|\vec{a}|^2=8$

(4) $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$

$\Rightarrow \vec{a},\vec{b},\vec{c}$ are mutually $\perp$ vectors.

$\therefore |\vec{a}\times\vec{b}|=|\vec{c}|\Rightarrow|\vec{a}||\vec{b}|=|\vec{c}|\Rightarrow|\vec{b}|=|\vec{c}|/2$

Also, $|\vec{b}\times\vec{c}|=|\vec{a}|\Rightarrow|\vec{b}||\vec{c}|=2\ \Rightarrow |\vec{c}|=2\ \&\ |\vec{b}|=1$

$|3a+b-2c|^2=(3\vec{a}+\vec{b}-2\vec{c})\cdot(3\vec{a}+\vec{b}-2\vec{c})$

$=9|\vec{a}|^2+|\vec{b}|^2+4|\vec{c}|^2$

$=(9\times 4)+1+(4\times 4)$

$=36+1+16=53$

9. The values of λ and μ such that the system of equations $x+y+z=6$, $3x+5y+5z=26$, $x+2y+\lambda z=\mu$ has no solution, are :

(1) $\lambda=3,\mu=5$ (2) $\lambda=3,\mu\neq 10$

(3) $\lambda\neq 2,\mu=10$ (4) $\lambda=2,\mu\neq 10$

Official Ans. by NTA (4)

Sol. $x + y + z = 6$ ...(i)

$3x + 5y + 5z = 26$ ...(ii)

$x + 2y + \lambda z = \mu$ ...(iii)

$5 \times (i) - (ii) \Rightarrow 2x = 4 \Rightarrow x = 2$

$\therefore$ from (i) and (iii)

$y + z = 4$ ...(iv)

$2y + \lambda z = \mu - 2$ ...(v)

(v) – 2 × (iv)

$\Rightarrow (\lambda - 2)z = \mu - 10$

$\Rightarrow z = \dfrac{\mu - 10}{\lambda - 2}$ & $y = 4 - \dfrac{\mu - 10}{\lambda - 2}$

$\therefore$ For no solution $\lambda = 2$ and $\mu \neq 10$.

10. If the shortest distance between the straight lines $3(x - 1) = 6(y - 2) = 2(z - 1)$ and

$4(x - 2) = 2(y - \lambda) = (z - 3),\ \lambda \in \mathbf{R}$ is $\dfrac{1}{\sqrt{38}}$, then

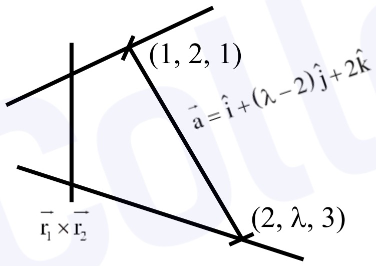
the integral value of λ is equal to :

(1) 3 (2) 2 (3) 5 (4) –1

Official Ans. by NTA (1)

Sol. $L_1: \dfrac{(x-1)}{2} = \dfrac{(y-2)}{1} = \dfrac{(z-1)}{3}$ $\quad \vec{r}_1 = 2\hat{i} + \hat{j} + 3\hat{k}$

$L_2: \dfrac{(x-2)}{1} = \dfrac{y-\lambda}{2} = \dfrac{z-3}{4}$ $\quad \vec{r}_2 = \hat{i} + 2\hat{j} + 4\hat{k}$



Shortest distance = Projection of $\vec{a}$ on $\vec{r}_1 \times \vec{r}_2$

$= \dfrac{\left|\vec{a}.\left(\vec{r}_1 \times \vec{r}_2\right)\right|}{\left|\vec{r}_1 \times \vec{r}_2\right|}$

$\left|\vec{a} \cdot \left(\vec{r}_1 \times \vec{r}_2\right)\right| = \begin{vmatrix} 1 & \lambda - 2 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = |14 - 5\lambda|$

$\left|\vec{r}_1 \times \vec{r}_2\right| = \sqrt{38}$

$\therefore \dfrac{1}{\sqrt{38}} = \dfrac{|14 - 5\lambda|}{\sqrt{38}}$

$\Rightarrow |14 - 5\lambda| = 1$

$\Rightarrow 14 - 5\lambda = 1$ or $14 - 5\lambda = -1$

$\Rightarrow \lambda = \dfrac{13}{5}$ or 3

$\therefore$ Integral value of $\lambda = 3$.

11. Which of the following Boolean expressions is **not** a tautology ?

(1) $(p \Rightarrow q) \vee (\sim q \Rightarrow p)$

(2) $(q \Rightarrow p) \vee (\sim q \Rightarrow p)$

(3) $(p \Rightarrow \sim q) \vee (\sim q \Rightarrow p)$

(4) $(\sim p \Rightarrow q) \vee (\sim q \Rightarrow p)$

Official Ans. by NTA (4)

Sol. (1) $(p \rightarrow q) \vee (\sim q \rightarrow p)$

$= (\sim p \vee q) \vee (q \vee p)$

$= (\sim p \vee p) \vee q$

$= t \vee q = t$

(2) $(q \rightarrow p) \vee (\sim q \rightarrow p)$

$= (\sim q \vee p) \vee (q \vee p)$

$= (\sim q \vee q) \vee p$

$= t \vee p = t$

(3) $(p \rightarrow \sim q) \vee (\sim q \rightarrow p)$

$= (\sim p \vee \sim q) \vee (q \vee p)$

$= (\sim p \vee p) \vee (\sim q \vee q)$

$= t \vee t = t$

(4) $(\sim q \rightarrow q) \vee (\sim q \rightarrow p)$

$= (p \vee q) \vee (q \vee p)$

$= (p \vee p) \vee (q \vee p)$

$= p \vee q$

Which is not a tautology.

12. Let $A = [a_{ij}]$ be a real matrix of order 3 × 3, such that $a_{i1} + a_{i2} + a_{i3} = 1$, for i = 1, 2, 3. Then, the sum of all the entries of the matrix A^3 is equal to :

(1) 2 (2) 1 (3) 3 (4) 9

Official Ans. by NTA (3)

Sol. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$AX = \begin{bmatrix} a_{11}+a_{12}+a_{13} \\ a_{21}+a_{22}+a_{23} \\ a_{31}+a_{32}+a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow AX = X$

Replace X by AX

$A^2X = AX = X$

Replace X by AX

$A^3X = AX = X$

$$\text{Let } A^3 = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$A^3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1+x_2+x_3 \\ y_1+y_2+y_3 \\ z_1+z_2+z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Sum of all the element = 3

13. Let [x] denote the greatest integer less than or equal to x. Then, the values of $x \in \mathbf{R}$ satisfying the equation $[e^x]^2 + [e^x + 1] - 3 = 0$ lie in the interval :

(1) $\left[0, \frac{1}{e}\right)$ (2) $[\log_e 2, \log_e 3)$

(3) $[1, e)$ (4) $[0, \log_e 2)$

Official Ans. by NTA (4)

Sol. $\left[e^x\right]^2 + \left[e^x + 1\right] - 3 = 0$

$\Rightarrow \left[e^x\right]^2 + \left[e^x\right] + 1 - 3 = 0$

Let $\left[e^x\right] = t$

$\Rightarrow t^2 + t - 2 = 0$

$\Rightarrow t = -2, 1$

$\left[e^x\right] = -2$ (Not possible)

or $\left[e^x\right] = 1 \quad \therefore \ 1 \le e^x < 2$

$\Rightarrow \ \ln(1) \le x < \ln(2)$

$\Rightarrow \ 0 \le x < \ln(2)$

$\Rightarrow \ x \in [0, \ln 2)$

14. Let the circle $S : 36x^2 + 36y^2 - 108x + 120y + C = 0$ be such that it neither intersects nor touches the co-ordinate axes. If the point of intersection of the lines, $x - 2y = 4$ and $2x - y = 5$ lies inside the circle S, then :

(1) $\frac{25}{9} < C < \frac{13}{3}$ (2) $100 < C < 165$

(3) $81 < C < 156$ (4) $100 < C < 156$

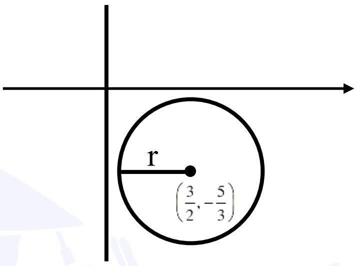
Official Ans. by NTA (4)

Sol. $S : 36x^2 + 36y^2 - 108x + 120y + C = 0$

$\Rightarrow x^2 + y^2 - 3x + \frac{10}{3}y + \frac{C}{36} = 0$

Centre $\equiv (-g, -f) \equiv \left(\frac{3}{2}, \frac{-10}{6}\right)$

radius $= r = \sqrt{\frac{9}{4} + \frac{100}{36} - \frac{C}{36}}$



Now,

$\Rightarrow r < \frac{3}{2}$

$\Rightarrow \frac{9}{4} + \frac{100}{36} - \frac{C}{36} < \frac{9}{4}$

$\Rightarrow C > 100$ .....(1)

Now point of intersection of $x - 2y = 4$ and $2x - y = 5$ is $(2, -1)$, which lies inside the circle S.

$\therefore \ S(2, -1) < 0$

$\Rightarrow (2)^2 + (-1)^2 - 3(2) + \frac{10}{3}(-1) + \frac{C}{36} < 0$

$\Rightarrow 4 + 1 - 6 - \frac{10}{3} + \frac{C}{36} < 0$

$\boxed{C < 156}$ .....(2)

From (1) & (2)

$\boxed{100 < C < 156}$ Ans.

15. Let n denote the number of solutions of the equation $z^2 + 3\bar{z} = 0$, where z is a complex number. Then the value of $\sum_{k=0}^{\infty} \frac{1}{n^k}$ is equal to

(1) 1 (2) $\frac{4}{3}$ (3) $\frac{3}{2}$ (4) 2

Official Ans. by NTA (2)

Sol. $z^2 + 3\bar{z} = 0$

Put $z = x + iy$

$\Rightarrow x^2 - y^2 + 2ixy + 3(x - iy) = 0$

$\Rightarrow (x^2 - y^2 + 3x) + i(2xy - 3y) = 0 + i0$

$\therefore\ x^2 - y^2 + 3x = 0$ .....(1)

$2xy - 3y = 0$ .....(2)

$x = \frac{3}{2}, y = 0$

Put $x = \frac{3}{2}$ in equation (1)

$\frac{9}{4} - y^2 + \frac{9}{2} = 0$

$y^2 = \frac{27}{4} \Rightarrow y = \pm\frac{3\sqrt{3}}{2}$

$\therefore\ (x, y) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right), \left(\frac{3}{2}, \frac{-3\sqrt{3}}{2}\right)$

Put $y = 0 \Rightarrow x^2 - 0 + 3x = 0$

$x = 0, -3$

$\therefore\ (x, y) = (0, 0), (-3, 0)$

$\therefore$ No of solutions = n = 4

$\sum_{K=0}^{\infty}\left(\frac{1}{n^k}\right) = \sum_{K=0}^{\infty}\left(\frac{1}{4^k}\right)$

$= \frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ...$

$= \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$

16. The number of solutions of $\sin^7 x + \cos^7 x = 1$, $x \in [0, 4\pi]$ is equal to

(1) 11 (2) 7 (3) 5 (4) 9

Official Ans. by NTA (3)

Sol. $\sin^7 x \le \sin^2 x \le 1$ ...(1)

and $\cos^7 x \le \cos^2 x \le 1$ ...(2)

also $\sin^2 x + \cos^2 x = 1$

$\Rightarrow$ equality must hold for (1) & (2)

$\Rightarrow \sin^7 x = \sin^2 x$ & $\cos^7 = \cos^2 x$

$\Rightarrow \sin x = 0$ & $\cos x = 1$

or

$\cos x = 0$ & $\sin x = 1$

$\Rightarrow x = 0, 2\pi, 4\pi, \frac{\pi}{2}, \frac{5\pi}{2}$

$\Rightarrow$ 5 solutions

17. If the domain of the function $f(x) = \frac{\cos^{-1}\sqrt{x^2 - x + 1}}{\sqrt{\sin^{-1}\left(\frac{2x-1}{2}\right)}}$ is the interval $(\alpha, \beta]$, then $\alpha + \beta$ is equal to :

(1) $\frac{3}{2}$ (2) 2 (3) $\frac{1}{2}$ (4) 1

Official Ans. by NTA (1)

Sol. $O \le x^2 - x + 1 \le 1$

$\Rightarrow x^2 - x \le 0$

$\Rightarrow x \in [0, 1]$

Also, $0 < \sin^{-1}\left(\frac{2x-1}{2}\right) \le \frac{\pi}{2}$

$\Rightarrow 0 < \frac{2x-1}{2} \le 1$

$\Rightarrow 0 < 2x - 1 \le 2$

$1 < 2x \le 3$

$\frac{1}{2} < x \le \frac{3}{2}$

Taking intersection

$x \in \left(\frac{1}{2}, 1\right]$

$\Rightarrow \alpha = \frac{1}{2}, \beta = 1$

$\Rightarrow \alpha + \beta = \frac{3}{2}$

18. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as

$$f(x) = \begin{cases} \frac{x^3}{(1 - \cos 2x)^2}\log_e\left(\frac{1 + 2xe^{-2x}}{(1 - xe^{-x})^2}\right) & , \quad x \ne 0 \\ \alpha & , \quad x = 0 \end{cases}.$$

If f is continuous at $x = 0$, then α is equal to :

(1) 1 (2) 3 (3) 0 (4) 2

Official Ans. by NTA (1)

Sol. For continuity

$\lim_{x \to 0} \frac{x^3}{4\sin^4 x}\left(\ell n(1 + 2xe^{-2x}) - 2\ell n(1 - xe^{-x})\right)$

$= \alpha$

$\lim_{x \to 0} \frac{1}{4x}\left[2xe^{-2x} + 2xe^{-x}\right] = \alpha$

$= \frac{1}{4}(4) = \alpha = 1$

19. Let a line L : $2x + y = k, k > 0$ be a tangent to the hyperbola $x^2 - y^2 = 3$. If L is also a tangent to the parabola $y^2 = \alpha x$, then α is equal to :

(1) 12 (2) –12 (3) 24 (4) –24

Official Ans. by NTA (4)

Sol. Tangent to hyperbola of

Slope m = –2 (given)

$$y = -2x \pm \sqrt{3(3)}$$

$$\left(y = mx \pm \sqrt{a^2m^2 - b^2}\right)$$

$$\Rightarrow y + 2x = \pm 3 \Rightarrow 2x + y = 3 \quad (k > 0)$$

For parabola $y^2 = \alpha x$

$$y = mx + \frac{\alpha}{4m}$$

$$\Rightarrow y = -2x + \frac{\alpha}{-8} \Rightarrow \frac{\alpha}{-8} = 3$$

$$\Rightarrow \alpha = -24.$$

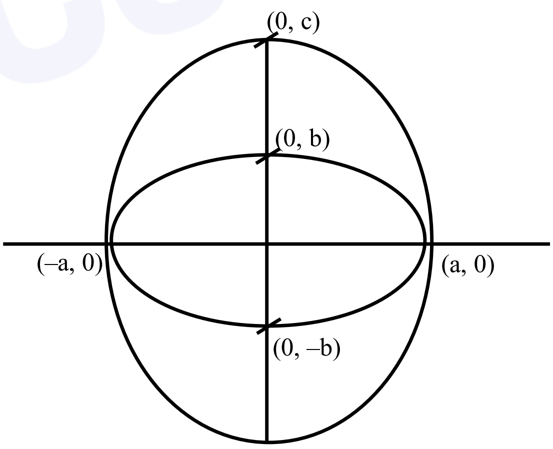
20. Let $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a > b. Let E_2 be another ellipse such that it touches the end points of major axis of E_1 and the foci of E_2 are the end points of minor axis of E_1. If E_1 and E_2 have same eccentricities, then its value is :

(1) $\frac{-1+\sqrt{5}}{2}$ (2) $\frac{-1+\sqrt{8}}{2}$

(3) $\frac{-1+\sqrt{3}}{2}$ (4) $\frac{-1+\sqrt{6}}{2}$

Official Ans. by NTA (1)

Sol.



$$e^2 = 1 - \frac{b^2}{a^2}$$

$$e^2 = 1 - \frac{a^2}{c^2}$$

$$\Rightarrow \frac{b^2}{a^2} = \frac{a^2}{c^2}$$

$$\Rightarrow c^2 = \frac{a^4}{b^2} \Rightarrow c = \frac{a^2}{b}$$

Also b = ce

$$\Rightarrow c = \frac{b}{e}$$

$$\frac{b}{e} = \frac{a^2}{b}$$

$$\Rightarrow e = \frac{b^2}{a^2} = 1 - e^2$$

$$\Rightarrow e^2 + e - 1 = 0$$

$$\Rightarrow e = \frac{-1+\sqrt{5}}{2}$$

SECTION-B

1. Let A = {0, 1, 2, 3, 4, 5, 6, 7}. Then the number of bijective functions $f : A \to A$ such that $f(1) + f(2) = 3 - f(3)$ is equal to

Official Ans. by NTA (720)

Sol. $f(1) + f(2) = 3 - f(3)$

$$\Rightarrow f(1) + f(2) = 3 + f(3) = 3$$

The only possibility is : 0 + 1 + 2 = 3

⇒ Elements 1, 2, 3 in the domain can be mapped with 0, 1, 2 only.

So number of bijective functions.

$$= \lfloor 3 \times \lfloor 5 = 720$$

2. If the digits are not allowed to repeat in any number formed by using the digits 0, 2, 4, 6, 8, then the number of all numbers greater than 10,000 is equal to ________.

Official Ans. by NTA (96)

Sol.

2,4,6,8				
4	4	3	2	1

$$= 4 \times 4 \times 3 \times 2 = 96$$

3. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then the number of 3 × 3 matrices B with entries from the set {1, 2, 3, 4, 5} and satisfying AB = BA is _______.

Official Ans. by NTA (3125)

Sol. Let matrix $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & n & i \end{bmatrix}$

$\because$ AB = BA

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} = \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix}$$

$\Rightarrow$ d = b, e = a, f = c, g=h

$\therefore$ Matrix $B = \begin{bmatrix} a & b & c \\ b & a & c \\ g & g & i \end{bmatrix}$

No. of ways of selecting a, b, c, g, i

= 5 × 5 × 5 × 5 × 5

= 5^5 = 3125

$\therefore$ No. of Matrices B = 3125

4. Consider the following frequency distribution :

Class :	0 – 6	6 – 12	12 – 18	18 – 24	24 – 30
Frequency :	a	b	12	9	5

If mean = $\frac{309}{22}$ and median = 14, then the value $(a - b)^2$ is equal to _______.

Official Ans. by NTA (4)

Sol.

Class	Frequency	x_i	$f_i\, x_i$
0-6	a	3	3a
6-12	b	9	9b
12-18	12	15	180
18-24	9	21	189
24-30	5	27	135
	N=(26+a+b)		**(504+3a+9b)**

Mean = $\frac{3a+9b+180+189+135}{a+b+26} = \frac{309}{22}$

$\Rightarrow$ 66a + 198b + 11088 = 309a + 309b + 8034

$\Rightarrow$ 243a + 111b = 3054

$\Rightarrow \boxed{81a + 37b = 1018} \rightarrow (1)$

Now, Median = $12 + \frac{\frac{a+b+26}{2} - (a+b)}{12} \times 6 = 14$

$\Rightarrow \frac{13}{2} - \left(\frac{a+b}{4}\right) = 2$

$\Rightarrow \frac{a+b}{4} = \frac{9}{2}$

$\Rightarrow \boxed{a+b=18} \rightarrow (2)$

From equation (1) & (2)

a = 8, b = 10

$\therefore (a - b)^2 = (8 - 10)^2$

5. The sum of all the elements in the set $\{n \in \{1, 2, \ldots, 100\} \mid$ H.C.F. of n and 2040 is 1$\}$ is equal to _______.

Official Ans. by NTA (1251)

Sol. $2040 = 2^3 \times 3 \times 5 \times 17$

n should not be multiple of 2, 3, 5 and 17.

Sum of all n = (1 + 3 + 5 ….. + 99) – (3 + 9 + 15 + 21 + ….. + 99) – (5 + 25 + 35 + 55 + 65 + 85 + 95) – (17)

$= 2500 - \frac{17}{2}(3+99) - 365 - 17$

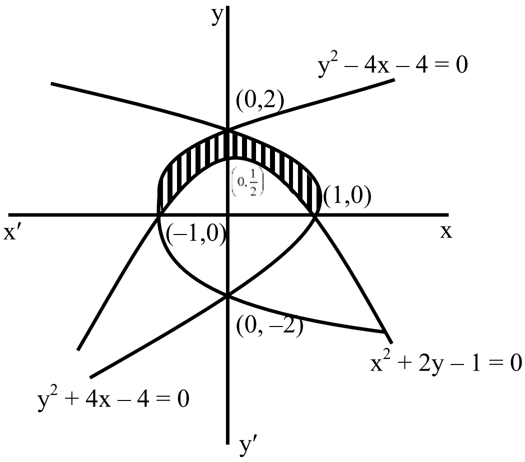
= 2500 – 867 – 365 – 17

= 1251

6. The area (in sq. units) of the region bounded by the curves $x^2 + 2y - 1 = 0$, $y^2 + 4x - 4 = 0$ and $y^2 - 4x - 4 = 0$, in the upper half plane is _______.

Official Ans. by NTA (2)

Sol.



Required Area (shaded)

$$= 2\left[\int_0^2\left(\frac{4-y^2}{4}\right)dy - \int_0^1\left(\frac{1-x^2}{2}\right)dx\right]$$

$$= 2\left[\frac{4}{3}-\frac{1}{3}\right] = (2)$$

7. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined as

$$f(x) = \begin{cases} 3\left(1-\frac{|x|}{2}\right) & \text{if } |x| \le 2 \\ 0 & \text{if } |x| > 2 \end{cases}$$

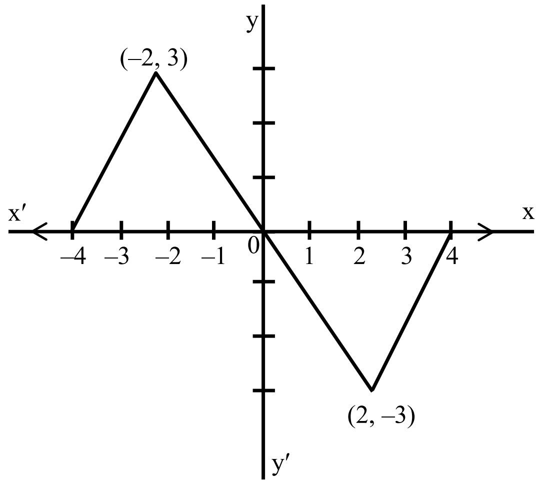
Let $g : \mathbf{R} \to \mathbf{R}$ be given by $g(x) = f(x+2) - f(x-2)$. If n and m denote the number of points in **R** where g is not continuous and not differentiable, respectively, then n + m is equal to ________.

Official Ans. by NTA (4)

Sol. $$f(x-2)\begin{cases} \frac{3x}{2} & -4 \le x \le -2 \\ -\frac{3x}{2} & -2 < x \le 0 \\ 0 & x \in (-\infty,-4)\cup(0,+\infty) \end{cases}$$

$$f(x-2)\begin{cases} \frac{3x}{2} & 0 \le x \le 2 \\ -\frac{3x}{2}+6 & 2 \le x \le 4 \\ 0 & x \in (-\infty,0)\cup(4,+\infty) \end{cases}$$

$$g(x) = f(x+2) - f(x-2)\begin{cases} \frac{3x}{2}+6 & -4 \le x \le -2 \\ -\frac{3x}{2} & -2<x<2 \\ \frac{3x}{2}-6 & 2 \le x \le 4 \end{cases}$$



n = 0

m = 4 ⇒ (n + m = 4)

8. If the constant term, in binomial expansion of $\left(2x^r + \frac{1}{x^2}\right)^{10}$ is 180, then r is equal to ________.

Official Ans. by NTA (8)

Sol.

$$\left(2x^r + \frac{1}{x^2}\right)^{10}$$

General term = ${}^{10}C_R\left(2x^2\right)^{10-R}x^{-2R}$

$\Rightarrow 2^{10-R}\,{}^{10}C_R = 180$ …….. (1)

& (10 – R) r – 2R = 0

$$r = \frac{2R}{10-R}$$

$$r = \frac{2(R-10)}{10-R} + \frac{20}{10-R}$$

$$\Rightarrow r = -2 + \frac{20}{10-R} \quad \ldots\ldots (2)$$

R = 8 or 5 reject equation (1) not satisfied

At R = 8

$2^{10-R}\,{}^{10}C_R = 180 \Rightarrow \boxed{r=8}$

9. Let y = y(x) be the solution of the differential equation $\left((x+2)e^{\left(\frac{y+1}{x+2}\right)} + (y+1)\right)$ dx = (x + 2) dy, y(1) = 1. If the domain of y = y(x) is an open interval (α, β), then |α + β| is equal to ________.

Official Ans. by NTA (1 or 4)

Sol. $y + 1 = Y \Rightarrow dy = dY$

$x + 2 = X \Rightarrow dx = dX$

$$\Rightarrow \left(Xe^{\frac{Y}{X}} + Y\right)dX = XdY$$

$$\Rightarrow XdY - YdX = Xe^{Y/X}dX$$

$$\Rightarrow d\left(\frac{Y}{X}\right)e^{-\frac{Y}{X}} = \frac{dX}{X}$$

$$-e^{-Y/X} = \ell|X| + c$$

$$(3, 2) \rightarrow -e^{-2/3} = \ell|3| + c$$

$$-e^{-\frac{Y}{X}} = \ell n|X| - e^{-\frac{2}{3}} - \ell n3$$

$$e^{-\frac{Y}{X}} = e^{2/3} + \ell n3 - \ell n|X| > 0$$

$$\ell n|X| < \left(e^{2/3} + \ell n3\right)$$

Let $\lambda = \left(e^{2/3} + \ell n3\right)$

$|x + 2| < e^{\lambda}$

$-e^{\lambda} < x + 2 < e^{\lambda}$

$-e^{\lambda} - 2 < x < e^{\lambda} - 2$

$\alpha \qquad \beta$

$\alpha + \beta = -4 \Rightarrow |\alpha + \beta| = 4$

Although $x = -2$ should be excluded from domain but according to the given problem it will the most appropriate solution.

10. The number of elements in the set $\{n \in \{1, 2, 3, \ldots, 100\} \mid (11)^n > (10)^n + (9)^n\}$ is ________.

Official Ans. by NTA (96)

Sol. $11^n > 10^n + 9^n$

$\Rightarrow 11^n - 9^n > 10^n$

$\Rightarrow (10 + 1)^n - (10 - 1)^n > 10^n$

$\Rightarrow \left\{ {}^nC_1 \cdot 10^{n-1} + {}^nC_3 10^{n-0} + {}^nC_5 10^{n-5} + \cdots\right\} > 10^n$

$\Rightarrow 2n \cdot 10^{n-1} + 2\left\{{}^nC_3 10^{n-3} + {}^nC_5 10^{n-5} + \cdots\right\} > 10^n$

..... (1)

For $n = 5$

$10^5 + 2\left\{{}^5C_3 10^2 + {}^5C_5\right\} > 10^5$ (True)

For $n = 6, 7, 8, \ldots 100$

$2n10^{n-1} > 10^n$

$\Rightarrow 2n10^{n-1} + 2\left\{{}^nC_3 10^{n-3} + {}^nC_5 10^{n-5} + \cdots\right\} > 10^n$

$\Rightarrow 11^n - 9^n > 10^n$ For $n = 5, 6, 7, \ldots 100$

For $n = 4$, Inequality (1) is not satisfied

$\Rightarrow$ Inequality does not hold good for

$N = 1, 2, 3, 4$

So, required number of elements

$= 96$