FINAL JEE–MAIN EXAMINATION – FEBRUARY, 2021

(Held On Thursday 25th February, 2021) TIME : 9 : 00 AM to 12 : 00 NOON

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. When a missile is fired from a ship, the probability that it is intercepted is $\frac{1}{3}$ and the probability that the missile hits the target, given that it is not intercepted, is $\frac{3}{4}$. If three missiles are fired independently from the ship, then the probability that all three hit the target, is :

(1) $\frac{1}{27}$ (2) $\frac{3}{4}$ (3) $\frac{1}{8}$ (4) $\frac{3}{8}$

Official Ans. by NTA (3)

Sol. Required probability $= \left(\frac{2}{3}\times\frac{3}{4}\right)^3 = \frac{1}{8}$

2. If $0<\theta,\phi<\frac{\pi}{2}, x=\sum_{n=0}^{\infty}\cos^{2n}\theta,\ y=\sum_{n=0}^{\infty}\sin^{2n}\phi$

and $z=\sum_{n=0}^{\infty}\cos^{2n}\theta\cdot\sin^{2n}\phi$ then :

(1) $xy - z = (x + y) z$ (2) $xy + yz + zx = z$

(3) $xyz = 4$ (4) $xy + z = (x + y)z$

Official Ans. by NTA (4)

Sol. $x = \frac{1}{1-\cos^2\theta} \quad \Rightarrow \sin^2\theta = \frac{1}{x}$

Also, $\cos^2\theta = \frac{1}{y}$ & $1-\sin^2\theta\cos^2\theta = \frac{1}{z}$

So, $1-\frac{1}{x}\times\frac{1}{y}=\frac{1}{z} \Rightarrow z(xy-1)=xy$ ....(1)

Also, $\frac{1}{x}+\frac{1}{y}=1 \quad \Rightarrow x+y=xy$ .....(2)

From (i) and (ii)

$xy + z = xyz = (x + y) z$

3. Let $f, g : N \to N$ such that $f(n + 1) = f(n) + f(1)$ $\forall\ n \in N$ and g be any arbitrary function. Which of the following statements is NOT true?

(1) If fog is one-one, then g is one-one

(2) If f is onto, then $f(n) = n \ \forall n\in N$

(3) f is one-one

(4) If g is onto, then fog is one-one

Official Ans. by NTA (4)

Sol. $f(n + 1) - f(n) = f(1)$

$\Rightarrow f(n) = nf(1)$

$\Rightarrow$ f is one-one

Now, Let $f(g(x_2)) = f(g(x_1))$

$\Rightarrow g(x_2) = g(x_1)$ (as f is one-one)

$\Rightarrow x_1 = x_2$ (as fog is one-one)

$\Rightarrow$ g is one-one

Now, $f(g(n)) = g(n) f(1)$

may be many-one if

g(n) is many-one

4. The equation of the line through the point (0,1,2) and perpendicular to the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{-2}$ is :

(1) $\frac{x}{3}=\frac{y-1}{4}=\frac{z-2}{3}$

(2) $\frac{x}{3}=\frac{y-1}{-4}=\frac{z-2}{3}$

(3) $\frac{x}{3}=\frac{y-1}{4}=\frac{z-2}{-3}$

(4) $\frac{x}{-3}=\frac{y-1}{4}=\frac{z-2}{3}$

Official Ans. by NTA (4)

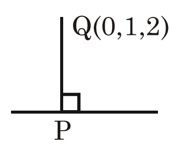
Sol. $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{-2}=r$

$\Rightarrow P(x, y, z) = (2r + 1, 3r - 1, -2r + 1)$

Since, $\overrightarrow{QP}\perp(2\hat{i}+3\hat{j}-2\hat{k})$

$\Rightarrow 4r + 2 + 9r - 6 + 4r + 2 = 0$

$\Rightarrow r=\frac{2}{17}$



$\Rightarrow P\left(\frac{21}{17},\frac{-11}{17},\frac{13}{17}\right)$

$\Rightarrow \overrightarrow{PQ}=\frac{21\hat{i}-28\hat{j}-21\hat{k}}{17}$

So, $\overrightarrow{QP}: \frac{x}{-3}=\frac{y-1}{4}=\frac{z-2}{3}$

5. Let α be the angle between the lines whose direction cosines satisfy the equations $l + m - n = 0$ and $l^2 + m^2 - n^2 = 0$. Then the value of $\sin^4\alpha + \cos^4\alpha$ is :

(1) $\frac{3}{4}$ (2) $\frac{3}{8}$ (3) $\frac{5}{8}$ (4) $\frac{1}{2}$

Official Ans. by NTA (3)

Sol. $n = \ell + m$

Now, $\ell^2 + m^2 = n^2 = (\ell + m)^2$

$\Rightarrow 2\ell m = 0$

If $\ell = 0 \Rightarrow m = n = \pm\frac{1}{\sqrt{2}}$

And, If $m = 0 \Rightarrow n = \ell = \pm\frac{1}{\sqrt{2}}$

So, direction cosines of two lines are

$\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

Thus, $\cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$

6. The value of the integral

$$\int \frac{\sin\theta.\sin 2\theta(\sin^6\theta + \sin^4\theta + \sin^2\theta)\sqrt{2\sin^4\theta + 3\sin^2\theta + 6}}{1 - \cos 2\theta} d\theta$$

is :

(where c is a constant of integration)

(1) $\frac{1}{18}\left[11 - 18\sin^2\theta + 9\sin^4\theta - 2\sin^6\theta\right]^{\frac{3}{2}} + c$

(2) $\frac{1}{18}\left[9 - 2\cos^6\theta - 3\cos^4\theta - 6\cos^2\theta\right]^{\frac{3}{2}} + c$

(3) $\frac{1}{18}\left[9 - 2\sin^6\theta - 3\sin^4\theta - 6\sin^2\theta\right]^{\frac{3}{2}} + c$

(4) $\frac{1}{18}\left[11 - 18\cos^2\theta + 9\cos^4\theta - 2\cos^6\theta\right]^{\frac{3}{2}} + c$

Official Ans. by NTA (4)

Sol. $I = \int \frac{\sin\theta.\sin 2\theta(\sin^6\theta + \sin^4\theta + \sin^2\theta)\sqrt{2\sin^4\theta + 3\sin^2\theta + 6}}{1 - \cos 2\theta} d\theta$

$$\Rightarrow I = \int \frac{\sin\theta.2\sin\theta\cos\theta.\sin^2\theta(\sin^4\theta + \sin^2\theta + 1)(2\sin^4\theta + 3\sin^2\theta + 6)^{1/2}}{2\sin^2\theta} d\theta$$

$$= \int \sin^2\theta . \cos\theta(\sin^4\theta + \sin^2\theta + 1)(2\sin^4\theta + 3\sin^2\theta + 6)^{1/2} d\theta$$

$$\therefore I = \int t^2(t^4 + t^2 + 1)(2t^4 + 3t^2 + 6)^{1/2} dt$$

$$= \int (t^5 + t^3 + t)t(2t^4 + 3t^2 + 6)^{1/2} dt$$

$$= \int (t^5 + t^3 + t)(t^2)^{1/2}(2t^4 + 3t^2 + 6)^{1/2} dt$$

$$= \int (t^5 + t^3 + t)(2t^6 + 3t^4 + 6t^2)^{1/2} dt$$

Let $2t^6 + 3t^4 + 6t^2 = u^2$

$\Rightarrow 12(t^5 + t^3 + t)\, dt = 2udu$

$$\therefore I = \int (u^2)^{1/2} \cdot \frac{2udu}{12}$$

$$= \int \frac{u^2}{6} du = \frac{u^3}{18} + C$$

$$= \frac{(2t^6 + 3t^4 + 6t^2)^{3/2}}{18} + C$$

when $t = \sin\theta$

and $t^2 = 1 - \cos^2\theta$ will give option (4)

7. The value of $\int_{-1}^{1} x^2 e^{[x^3]} dx$, where [t] denotes the greatest integer $\leq$ t, is :

(1) $\frac{e-1}{3e}$ (2) $\frac{e+1}{3}$ (3) $\frac{e+1}{3e}$ (4) $\frac{1}{3e}$

Official Ans. by NTA (3)

Sol. $I = \int_{-1}^{1} x^2 e^{[x^3]} dx$

$$= \int_{-1}^{0} x^2 e^{[x^3]} dx + \int_{0}^{1} x^2 e^{[x^3]} dx$$

$$= \int_{-1}^{0} x^2 e^{-1} dx + \int_{0}^{1} x^2 e^{0} dx$$

$$= \frac{1}{e} \times \frac{x^3}{3}\Big|_{-1}^{0} + \frac{x^3}{3}\Big|_{0}^{1}$$

$$= \frac{1}{e} \times \left(0 - \left(\frac{-1}{3}\right)\right) + \frac{1}{3}$$

1 1 1+e

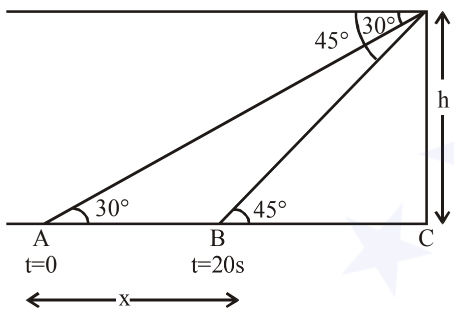
8. A man is observing, from the top of a tower, a boat speeding towards the tower from a certain point A, with uniform speed. At that point, angle of depression of the boat with the man's eye is 30° (Ignore man's height). After sailing for 20 seconds, towards the base of the tower (which is at the level of water), the boat has reached a point B, where the angle of depression is 45°. Then the time taken (in seconds) by the boat from B to reach the base of the tower is:

(1) 10 (2) $10\sqrt{3}$

(3) $10(\sqrt{3}+1)$ (4) $10(\sqrt{3}-1)$

Official Ans. by NTA (3)

Sol.



Let speed of boat is u m/s and height of tower is h meter & distance AB = x metre

$\therefore$ x = h cot 30° – h cot 45°

$\Rightarrow x = h(\sqrt{3}-1)$

$\therefore\ u = \frac{x}{20} = \frac{h(\sqrt{3}-1)}{20}\ m/s$

$\therefore$ Time taken to travel from B to C (Distance = h meter)

$$= \frac{h}{u} = \frac{h}{h\frac{(\sqrt{3}-1)}{20}} = \frac{20}{\sqrt{3}-1} = 10(\sqrt{3}+1)\sec.$$

9. A tangent is drawn to the parabola $y^2 = 6x$ which is perpendicular to the line 2x + y = 1. Which of the following points does NOT lie on it?

(1) (–6, 0) (2) (4, 5) (3) (5, 4) (4) (0, 3)

Official Ans. by NTA (3)

Sol. Slope of tangent = m_T = m

So, $m(-2) = -1 \Rightarrow m = \frac{1}{2}$

Equation : $y = mx + \frac{a}{m}$

$\Rightarrow y = \frac{1}{2}x + \frac{3}{2\times\frac{1}{2}}\left(a = \frac{6}{4} = \frac{3}{2}\right)$

$\Rightarrow y = \frac{x}{2} + 3$

$\Rightarrow$ 2y = x + 6

Point (5, 4) will not lie on it

10. All possible values of $\theta \in [0, 2\pi]$ for which $\sin 2\theta + \tan 2\theta > 0$ lie in :

(1) $\left(0,\frac{\pi}{2}\right)\cup\left(\pi,\frac{3\pi}{2}\right)$

(2) $\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)\cup\left(\pi,\frac{7\pi}{6}\right)$

(3) $\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)\cup\left(\frac{3\pi}{2},\frac{11\pi}{6}\right)$

(4) $\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)\cup\left(\pi,\frac{5\pi}{4}\right)\cup\left(\frac{3\pi}{2},\frac{7\pi}{4}\right)$

Official Ans. by NTA (4)

Sol. $\sin 2\theta + \tan 2\theta > 0$

$\Rightarrow \sin 2\theta + \frac{\sin 2\theta}{\cos 2\theta} > 0$

$\Rightarrow \sin 2\theta\frac{(\cos 2\theta + 1)}{\cos 2\theta} > 0 \Rightarrow \tan 2\theta(2\cos^2\theta) > 0$

Note : $\cos 2\theta \neq 0$

$\Rightarrow 1-2\sin^2\theta \neq 0 \Rightarrow \sin\theta \neq \pm\frac{1}{\sqrt{2}}$

Now, $\tan 2\theta\,(1 + \cos 2\theta) > 0$

$\Rightarrow \tan 2\theta > 0$ (as $\cos 2\theta + 1 > 0$)

$$\Rightarrow \theta \in \left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)\cup\left(\pi,\frac{5\pi}{4}\right)\cup\left(\frac{3\pi}{2},\frac{7\pi}{4}\right)$$

As $\sin\theta \neq \pm\frac{1}{\sqrt{2}}$; which has been already considered

11. Let the lines $(2 - i)z = (2 + i)\bar{z}$ and $(2 + i)z + (i - 2)\bar{z} - 4i = 0$, (here $i^2 = -1$) be normal to a circle C. If the line $iz + \bar{z} + 1 + i = 0$ is tangent to this circle C, then its radius is:

(1) $\frac{3}{\sqrt{2}}$ (2) $\frac{1}{2\sqrt{2}}$ (3) $3\sqrt{2}$ (4) $\frac{3}{2\sqrt{2}}$

Official Ans. by NTA (4)

Sol. (i) $(2 - i)z = (2 + i)\bar{z}$

$$\boxed{y=\frac{x}{2}}$$

(ii) $(2 + i)z + (i - 2)\bar{z} - 4i = 0$

$$\boxed{x+2y=2}$$

(iii) $iz+\bar{z}+1+i=0$

Eq^n of tangent $\boxed{x-y+1=0}$

Solving (i) and (ii)

$x = 1,\ y = \frac{1}{2}$

Now, $p = r \Rightarrow \left|\frac{1-\frac{1}{2}+1}{\sqrt{2}}\right| = r$

$\Rightarrow r = \frac{3}{2\sqrt{2}}$

12. The image of the point (3, 5) in the line $x - y + 1 = 0$, lies on :

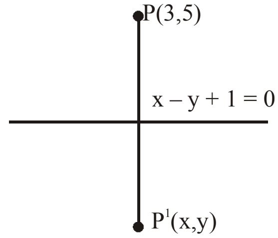
(1) $(x - 2)^2 + (y - 2)^2 = 12$

(2) $(x - 4)^2 + (y + 2)^2 = 16$

(3) $(x - 4)^2 + (y - 4)^2 = 8$

(4) $(x - 2)^2 + (y - 4)^2 = 4$

Sol.



$$\frac{x-3}{1}=\frac{y-5}{-1}=-2\left(\frac{3-5+1}{1+1}\right)$$

So, $x = 4, y = 4$

Hence, $(x - 2)^2 + (y - 4)^2 = 4$

13. If the curves, $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{c}+\frac{y^2}{d}=1$ intersect each other at an angle of 90°, then which of the following relations is TRUE?

(1) $a + b = c + d$

(2) $a - b = c - d$

(3) $a - c = b + d$

(4) $ab = \frac{c+d}{a+b}$

Official Ans. by NTA (2)

Sol. For orthogonal curves $a - c = b - d$

$\Rightarrow a - b = c - d$

14. $\lim\limits_{n\to\infty}\left(1+\frac{1+\frac{1}{2}+........+\frac{1}{n}}{n^2}\right)^n$ is equal to :

(1) $\frac{1}{2}$ (2) 0 (3) $\frac{1}{e}$ (4) 1

Official Ans. by NTA (4)

Sol. Given limit is of 1^∞ form

So, $l = \exp\left(\lim\limits_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{3}+........+\frac{1}{n}}{n}\right)$

Now,

$$0 \le 1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n} \le 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+....+\frac{1}{\sqrt{n}}$$

$$\le 2\sqrt{n}-1$$

So, $l = \exp(0)$ (from sandwich theorem)

15. The coefficients a, b and c of the quadratic equation, $ax^2 + bx + c = 0$ are obtained by throwing a dice three times. The probability that this equation has equal roots is:

(1) $\frac{1}{72}$ (2) $\frac{5}{216}$ (3) $\frac{1}{36}$ (4) $\frac{1}{54}$

Official Ans. by NTA (2)

Sol. $ax^2 + bx + c = 0$

For equal roots D = 0

$\Rightarrow b^2 = 4ac$

Case I : ac = 1

(a, b, c) = (1, 2, 1)

Case II : ac = 4

(a, b, c) = (1, 4, 4)

or (4, 4, 1)

or (2, 4, 2)

Case III : ac = 9

(a, b, c) = (3, 6, 3)

Required probability = $\frac{5}{216}$

16. The total number of positive integral solutions (x, y, z) such that xyz = 24 is :

(1) 36 (2) 24 (3) 45 (4) 30

Official Ans. by NTA (4)

Sol. $xyz = 2^3 \times 3^1$

Let $x = 2^{\alpha_1} \times 3^{\beta_1}$

$y = 2^{\alpha_2} \times 3^{\beta_2}$

$z = 2^{\alpha_3} \times 3^{\beta_2}$

Now $\alpha_1 + \alpha_2 + \alpha_3 = 3$.

No. of non-negative intergal sol = ${}^5C_2 = 10$

& $\beta_1 + \beta_2 + \beta_3 = 1$

No. of non-negative intergal sol^n = ${}^3C_2 = 3$

Total ways = 10 × 3 = 30.

17. The integer 'k', for which the inequality $x^2 - 2(3k - 1)x + 8k^2 - 7 > 0$ is valid for every x in R, is :

(1) 3 (2) 2 (3) 0 (4) 4

Official Ans. by NTA (1)

Sol. $x^2 - 2(3K - 1)x + 8K^2 - 7 > 0$

Now, D < 0

$\Rightarrow 4(3K - 1)^2 - 4 \times 1 \times (8K^2 - 7) < 0$

$\Rightarrow 9K^2 - 6K + 1 - 8K^2 + 7 < 0$

$\Rightarrow K^2 - 6K + 8 < 0$

$\Rightarrow (K - 4)(K - 2) < 0$

18. If a curve passes through the origin and the slope of the tangent to it at any point (x, y) is $\frac{x^2 - 4x + y + 8}{x - 2}$, then this curve also passes through the point:

(1) (5, 4) (2) (4, 5)

(3) (4, 4) (4) (5, 5)

Official Ans. by NTA (4)

Sol. Given

y (0) = 0

& $\frac{dy}{dx} = \frac{(x-2)^2 + y + 4}{x - 2}$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x-2} = (x-2) + \frac{4}{x-2}$

$\Rightarrow$ I.F. = $e^{-\int \frac{1}{x-2}dx} = \frac{1}{x-2}$

Solution of L.D.E.

$\Rightarrow y\frac{1}{x-2} = \int \frac{1}{x-2}\left((x-2) + \frac{4}{x-2}\right).dx$

$\Rightarrow \frac{y}{x-2} = x - \frac{4}{x-2} + C$

Now, at x = 0, y = 0 $\Rightarrow$ C = –2

y = x (x – 2) – 4 – 2 (x – 2)

$\Rightarrow y = x^2 - 4x$

This curve passes through (5, 5)

19. The statement $A \to (B \to A)$ is equivalent to :

(1) $A \to (A \wedge B)$ (2) $A \to (A \to B)$

(3) $A \to (A \leftrightarrow B)$ (4) $A \to (A \vee B)$

Official Ans. by NTA (4)

Sol. $A \to (B \to A)$

$\equiv A \to (\sim B \vee A)$

$\equiv \sim A \vee (\sim B \vee A)$

$\equiv (\sim A \vee A) \vee \sim B$

$\equiv T \vee \sim B \equiv T$

$\therefore T \vee B = T$

$\equiv (\sim A \vee A) \vee B$

$\equiv \sim A \vee (A \vee B)$

20. If Rolle's theorem holds for the function $f(x) = x^3 - ax^2 + bx - 4,\ x \in [1, 2]$ with $f'\left(\frac{4}{3}\right) = 0$, then ordered pair (a, b) is equal to :

(1) (5, 8) (2) (–5, 8)

(3) (5, –8) (4) (–5, –8)

Official Ans. by NTA (1)

Sol. $f(1) = f(2)$

$\Rightarrow 1 - a + b - 4 = 8 - 4a + 2b - 4$

$\Rightarrow 3a - b = 7$ ......(1)

Also $f^1\left(\frac{4}{3}\right) = 0$ (given)

$$\Rightarrow \left(3x^2 - 2ax + b\right)_{x=\frac{4}{3}} = 0$$

$$\Rightarrow \frac{16}{3} - \frac{8a}{3} + b = 0$$

$\Rightarrow 8a - 3b - 16 = 0$ ....(2)

Solving (1) and (2)

$a = 5,\ b = 8$

SECTION-B

1. Let f(x) be a polynomial of degree 6 in x, in which the coefficient of x^6 is unity and it has extrema at $x = -1$ and $x = 1$. If $\lim_{x\to 0}\frac{f(x)}{x^3} = 1$, then $5 \cdot f(2)$ is equal to _______.

Official Ans. by NTA (144)

Sol. Let $f(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

as $\lim_{x\to 0}\frac{f(x)}{x^3} = 1$ non-zero finite

So, $d = e = f = 0$

and $f(x) = x^3(x^3 + ax^2 + bx + c)$

Hence, $\lim_{x\to 0}\frac{f(x)}{x^3} = c = 1$

Now, as $f(x) = x^6 + ax^5 + bx^4 + x^3$

and $f'(x) = 0$ at $x = 1$ and $x = -1$

i.e., $f'(x) = 6x^5 + 5ax^4 + 4bx^3 + 3x^2$

$f'(1) = 0$

& $f'(-1) = 0$

$\Rightarrow -6 + 5a - 4b + 3 = 0$

$\Rightarrow 5a - 4b = 3$

Solving both we get,

$$a = \frac{-6}{10} = \frac{-3}{5};\quad b = \frac{-3}{2}$$

$$\therefore\ f(x) = x^6 - \frac{3}{5}x^5 - \frac{3}{2}x^4 + x^3$$

$$\therefore\ 5f(2) = 5\left[64 - \frac{3}{5}\cdot 32 - \frac{3}{2}\cdot 16 + 8\right]$$

$= 320 - 96 - 120 + 40$

$= 144$

2. The number of points, at which the function f(x) $= |2x + 1| - 3|x + 2| + |x^2 + x - 2|,\ x \in R$ is not differentiable, is _______.

Official Ans. by NTA (2)

Sol. $f(x) = |2x+1| - 3|x+2| + |x^2 + x - 2|$

$= |2x+1| - 3|x+2| + |x+2||x-1|$

$= |2x+1| + |x+2|\left(|x-1| - 3\right)$

Critical points are $x = \frac{-1}{2}$, –2, –1

but x = –2 is making a zero.

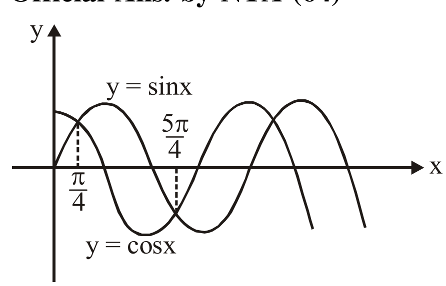
twice in product so, points of non differentability are $\boxed{x = \frac{-1}{2} \text{ and } x = -1}$

$\therefore$ Number of points of non-differentiability $= \boxed{2}$

3. The graphs of sine and cosine functions, intersect each other at a number of points and between two consecutive points of intersection, the two graphs enclose the same area A. Then A^4 is equal to ______.

Official Ans. by NTA (64)

Sol.



$$A = \int^{5\pi/4} \left(\sin x - \cos x\right) dx$$

$$= (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4}$$

$$= \left(-\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)\right) - \left(-\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\right)$$

$$\Rightarrow A = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = 2\sqrt{2}$$

$$\Rightarrow A^4 = \left(2\sqrt{2}\right)^4 = 16 \times 4 = 64$$

4. Let A_1, A_2, A_3, ... be squares such that for each $n \geq 1$, the length of the side of A_n equals the length of diagonal of A_{n+1}. If the length of A_1 is 12 cm, then the smallest value of n for which area of A_n is less than one, is ______.

Official Ans. by NTA (9)

Sol. Let a_n be the side length of A_n.

So, $a_n = \sqrt{2}a_{n+1}$, $a_1 = 12$

$$\Rightarrow a_n = 12 \times \left(\frac{1}{\sqrt{2}}\right)^{n-1}$$

Now, $(a_n)^2 < 1 \Rightarrow \frac{144}{2^{(n-1)}} < 1$

$\Rightarrow 2^{(n-1)} > 144$
$\Rightarrow n - 1 \geq 8$
$\Rightarrow n \geq 9$

5. Let $A = \begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix}$, where x, y and z are real numbers such that $x + y + z > 0$ and $xyz = 2$. If $A^2 = I_3$, then the value of $x^3 + y^3 + z^3$ is______.

Official Ans. by NTA (7)

Sol. $A^2 = I$

$\Rightarrow AA' = I \ (\text{as } A' = A)$

$\Rightarrow$ A is orthogonal

So, $x^2 + y^2 + z^2 = 1$ and $xy + yz + zx = 0$

$\Rightarrow (x + y + z)^2 = 1 + 2 \times 0$

$\Rightarrow x + y + z = 1$

Thus,

$x^3 + y^3 + z^3 = 3 \times 2 + 1 \times (1 - 0)$
$= 7$

6. If $A = \begin{bmatrix} 0 & -\tan\left(\frac{\theta}{2}\right) \\ \tan\left(\frac{\theta}{2}\right) & 0 \end{bmatrix}$ and

$(I_2 + A)(I_2 - A)^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then $13(a^2 + b^2)$ is equal to ________ .

Official Ans. by NTA (13)

Sol. $a^2 + b^2 = |I_2 + A||I_2 - A|^{-1}$

$$= \sec^2\frac{\theta}{2} \times \cos^2\frac{\theta}{2} = 1$$

7. The total number of numbers, lying between 100 and 1000 that can be formed with the digits 1, 2, 3, 4, 5, if the repetition of digits is not allowed and numbers are divisible by either 3 or 5, is ______.

Official Ans. by NTA (32)

Sol. We need three digits numbers.

Since $1 + 2 + 3 + 4 + 5 = 15$

So, number of possible triplets for multiple of 15 is $1 \times 2 \times 2$

so Ans. $= 4 \times \underline{|3} + 4 \times 3 - 1 \times 2 \times \underline{|2} = 32$

8. Let $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - \hat{j}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{a} = \vec{c} \times \vec{a}$ and $\vec{r} \cdot \vec{b} = 0$, then $\vec{r} \cdot \vec{a}$ is equal to _________.

Official Ans. by NTA (12)

Sol. $(\vec{r} - \vec{c}) \times \vec{a} = 0$

$\Rightarrow \vec{r} = \vec{c} + \lambda\vec{a}$

Now, $0 = \vec{b} \cdot \vec{c} + \lambda\vec{a} \cdot \vec{b}$

$$\Rightarrow \lambda = \frac{-\vec{b} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} = -\frac{2}{-1} = 2$$

9. If the system of equations

$kx + y + 2z = 1$

$3x - y - 2z = 2$

$-2x - 2y - 4z = 3$

has infinitely many solutions, then k is equal to ________.

Official Ans. by NTA (21)

Sol. We observe $5P_2 - P_1 = 3P_3$

So, $15 - K = -6$

$\Rightarrow K = 21$

10. The locus of the point of intersection of the lines

$(\sqrt{3})kx + ky - 4\sqrt{3} = 0$ and

$\sqrt{3}x - y - 4(\sqrt{3})k = 0$ is a conic, whose eccentricity is ___________.

Official Ans. by NTA (2)

Sol. $K = \frac{4\sqrt{3}}{\sqrt{3}x + y} = \frac{\sqrt{3}x - y}{4\sqrt{3}}$

$\Rightarrow 3x^2 - y^2 = 48$

$\Rightarrow \frac{x^2}{16} - \frac{y^2}{48} = 1$

Now, $48 = 16(e^2 - 1)$

$\Rightarrow \quad e = \sqrt{4} = 2$