MATHEMATICS

SECTION-A

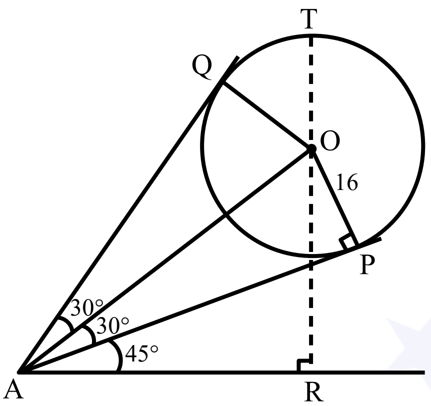
1. A spherical gas balloon of radius 16 meter subtends an angle 60° at the eye of the observer A while the angle of elevation of its center from the eye of A is 75°. Then the height (in meter) of the top most point of the balloon from the level of the observer's eye is :

(1) $8(2+2\sqrt{3}+\sqrt{2})$ (2) $8(\sqrt{6}+\sqrt{2}+2)$

(3) $8(\sqrt{2}+2+\sqrt{3})$ (4) $8(\sqrt{6}-\sqrt{2}+2)$

Official Ans. by NTA (2)

Sol.



O $\rightarrow$ centre of sphere

P,Q $\rightarrow$ point of contact of tangents from A

Let T be top most point of balloon & R be foot of perpendicular from O to ground.

From triangle OAP, OA = 16cosec30° = 32

From triangle ABO, $OR = OA \sin 75° = 32\dfrac{(\sqrt{3}+1)}{2\sqrt{2}}$

So level of top most point = OR + OT

$= 8\left(\sqrt{6}+\sqrt{2}+2\right)$

2. Let $f(x) = 3\sin^4 x + 10\sin^3 x + 6\sin^2 x - 3$, $x \in \left[-\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$. Then, f is :

(1) increasing in $\left(-\dfrac{\pi}{6}, \dfrac{\pi}{2}\right)$

(2) decreasing in $\left(0, \dfrac{\pi}{2}\right)$

(3) increasing in $\left(-\dfrac{\pi}{6}, 0\right)$

(4) decreasing in $\left(-\dfrac{\pi}{6}, 0\right)$

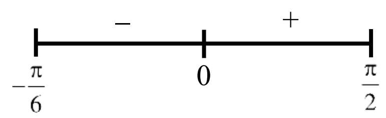
Official Ans. by NTA (4)

Sol. $f(x) = 3\sin^4 x + 10\sin^3 x + 6\sin^2 x - 3$, $x \in \left[-\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$

$f'(x) = 12\sin^3 x\cos x + 30\sin^2 x\cos x + 12\sin x\cos x$

$= 6\sin x\cos x\,(2\sin^2 x + 5\sin x + 2)$

$= 6\sin x\cos x\,(2\sin x + 1)(\sin + 2)$



Decreasing in $\left(-\dfrac{\pi}{6}, 0\right)$

3. Let S_n be the sum of the first n terms of an arithmetic progression. If $S_{3n} = 3S_{2n}$, then the value of $\dfrac{S_{4n}}{S_{2n}}$ is :

(1) 6 (2) 4 (3) 2 (4) 8

Official Ans. by NTA (1)

Sol. Let a be first term and d be common diff. of this A.P.

Given $S_{3n} = 3S_{2n}$

$\Rightarrow \dfrac{3n}{2}\left[2a+(3n-1)d\right] = 3\dfrac{2n}{2}\left[2a+(2n-1)d\right]$

$\Rightarrow 2a + (3n-1)d = 4a + (4n-2)d$

$\Rightarrow 2a + (n-1)d = 0$

Now $\dfrac{S_{4n}}{S_{2n}} = \dfrac{\frac{4n}{2}\left[2a+(4n-1)d\right]}{\frac{2n}{2}\left[2a+(2n-1)d\right]} = \dfrac{2\left[\underbrace{2a+(n-1)d}_{=0}+3nd\right]}{\left[\underbrace{2a+(n-1)d}_{=0}+nd\right]}$

$= \dfrac{6nd}{nd} = 6$

4. The locus of the centroid of the triangle formed by any point P on the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, and its foci is :

(1) $16x^2 - 9y^2 + 32x + 36y - 36 = 0$

(2) $9x^2 - 16y^2 + 36x + 32y - 144 = 0$

(3) $16x^2 - 9y^2 + 32x + 36y - 144 = 0$

(4) $9x^2 - 16y^2 + 36x + 32y - 36 = 0$

Official Ans. by NTA (1)

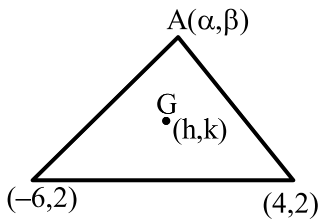
Sol. Given hyperbola is

$16(x+1)^2 - 9(y-2)^2 = 164 + 16 - 36 = 144$

$$\Rightarrow \frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$$

Eccentricity, $e = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$

$\Rightarrow$ foci are (4, 2) and (–6, 2)



Let the centroid be (h, k)

& A(α, β) be point on hyperbola

So $h = \frac{\alpha - 6 + 4}{3}, k = \frac{\beta + 2 + 2}{3}$

$\Rightarrow \alpha = 3h + 2, \beta = 3k - 4$

(α, β) lies on hyperbola so

$16(3h + 2 + 1)^2 - 9(3k - 4 - 2)^2 = 144$

$\Rightarrow 144(h+1)^2 - 81(k-2)^2 = 144$

$\Rightarrow 16(h^2 + 2h + 1) - 9(k^2 - 4k + 4) = 16$

$\Rightarrow 16x^2 - 9y^2 + 32x + 36y - 36 = 0$

5. Let the vectors

$(2+a+b)\hat{i} + (a+2b+c)\hat{j} - (b+c)\hat{k}$, $(1+b)\hat{i} + 2b\hat{j} - b\hat{k}$

and $(2+b)\hat{i} + 2b\hat{j} + (1-b)\hat{k}$ a, b, c, $\in$ **R**

be co-planar. Then which of the following is true?

(1) 2b = a + c (2) 3c = a + b

(3) a = b + 2c (4) 2a = b + c

Official Ans. by NTA (1)

Sol. If the vectors are co-planar,

$$\begin{vmatrix} a+b+2 & a+2b+c & -b-c \\ b+1 & 2b & -b \\ b+2 & 2b & 1-b \end{vmatrix} = 0$$

Now $R_3 \rightarrow R_3 - R_2$, $R_1 \rightarrow R_1 - R_2$

$$\text{So} \begin{vmatrix} a+1 & a+c & -c \\ b+1 & 2b & -b \\ 1 & 0 & 1 \end{vmatrix} = 0$$

= (a + 1) 2b – (a + c) (2b + 1) – c(–2b)

= 2ab + 2b – 2ab – a – 2bc – c + 2bc

= 2b – a – c = 0

6. Let f : **R** → **R** be defined as

$$f(x) = \begin{cases} \dfrac{\lambda|x^2 - 5x + 6|}{\mu(5x - x^2 - 6)}, & x < 2 \\ e^{\frac{\tan(x-2)}{x-[x]}} & , x > 2 \\ \mu & , x = 2 \end{cases}$$

where [x] is the greatest integer less than or equal to x. If f is continuous at x = 2, then λ + μ is equal to :

(1) e(–e + 1) (2) e(e – 2)

(3) 1 (4) 2e – 1

Official Ans. by NTA (1)

Sol. $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x \to 2^+} e^{\frac{\tan(x-2)}{x-2}} = e^1$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{-\lambda(x-2)(x-3)}{\mu(x-2)(x-3)} = -\frac{\lambda}{\mu}$$

For continuity $\mu = e = -\frac{\lambda}{\mu} \Rightarrow \mu = e, \lambda = -e^2$

λ + μ = e(–e + 1)

7. The value of the definite integral

$$\int_{\pi/24}^{5\pi/24} \frac{dx}{1 + \sqrt[3]{\tan 2x}} \text{ is :}$$

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{12}$ (4) $\frac{\pi}{18}$

Official Ans. by NTA (3)

Sol. Let $I = \int_{\pi/24}^{5\pi/24} \frac{(\cos 2x)^{1/3}}{(\cos 2x)^{1/3} + (\sin 2x)^{1/3}} dx$ ....(i)

$$\Rightarrow I = \int_{\pi/24}^{5\pi/24} \frac{\left(\cos\left\{2\left(\frac{\pi}{4} - x\right)\right\}\right)^{\frac{1}{3}}}{\left(\cos\left\{2\left(\frac{\pi}{4} - x\right)\right\}\right)^{\frac{1}{3}} + \left(\sin\left\{2\left(\frac{\pi}{4} - x\right)\right\}\right)^{\frac{1}{3}}} dx$$

$$\left\{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx\right\}$$

So $I = \int_{\pi/24}^{5\pi/24} \frac{(\sin 2x)^{1/3}}{(\sin 2x)^{1/3} + (\cos 2x)^{1/3}} dx$ ....(ii)

Hence $2I = \int_{\pi/24}^{5\pi/24} dx$ [(i) + (ii)]

$$\Rightarrow 2I = \frac{4\pi}{24} \Rightarrow \boxed{I = \frac{\pi}{12}}$$

8. If b is very small as compared to the value of a, so that the cube and other higher powers of $\frac{b}{a}$ can be neglected in the identity

$$\frac{1}{a-b}+\frac{1}{a-2b}+\frac{1}{a-3b}+....+\frac{1}{a-nb}=\alpha n+\beta n^2+\gamma n^3,$$

then the value of γ is :

(1) $\frac{a^2+b}{3a^3}$ (2) $\frac{a+b}{3a^2}$ (3) $\frac{b^2}{3a^3}$ (4) $\frac{a+b^2}{3a^3}$

Official Ans. by NTA (3)

Sol. $(a-b)^{-1}+(a-2b)^{-1}+....+(a-nb)^{-1}$

$$=\frac{1}{a}\sum_{r=1}^{n}\left(1-\frac{rb}{a}\right)^{-1}$$

$$=\frac{1}{a}\sum_{r=1}^{n}\left\{\left(1+\frac{rb}{a}+\frac{r^2b^2}{a^2}\right)+\left(\text{terms to be neglected}\right)\right\}$$

$$=\frac{1}{a}\left[n+\frac{n(n+1)}{2}.\frac{b}{a}+\frac{n(n+1)(2n+1)}{6}.\frac{b^2}{a^2}\right]$$

$$=\frac{1}{a}\left[n^3\left(\frac{b^2}{3a^2}\right)+....\right]$$

So $\gamma=\frac{b^2}{3a^3}$

9. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx}=1+xe^{y-x},-\sqrt{2}<x<\sqrt{2},y(0)=0$ then, the minimum value of $y(x)$, $x\in\left(-\sqrt{2},\sqrt{2}\right)$ is equal to :

(1) $\left(2-\sqrt{3}\right)-\log_e 2$

(2) $\left(2+\sqrt{3}\right)+\log_e 2$

(3) $\left(1+\sqrt{3}\right)-\log_e\left(\sqrt{3}-1\right)$

(4) $\left(1-\sqrt{3}\right)-\log_e\left(\sqrt{3}-1\right)$

Official Ans. by NTA (4)

Sol. $\frac{dy-dx}{e^{y-x}}=x\,dx$

$$\Rightarrow\frac{dy-dx}{e^{y-x}}=x\,dx$$

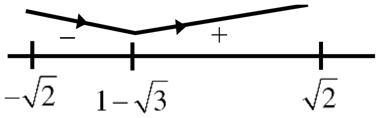
$$\Rightarrow -e^{x-y}=\frac{x^2}{2}+c$$

At $x = 0, y = 0 \Rightarrow c = -1$

$$\Rightarrow e^{x-y}=\frac{2-x^2}{2}$$

$$\Rightarrow y=x-\ell n\left(\frac{2-x^2}{2}\right)$$

$$\Rightarrow\frac{dy}{dx}=1+\frac{2x}{2-x^2}=\frac{2+2x-x^2}{2-x^2}$$



So minimum value occurs at $x = 1-\sqrt{3}$

$$y\left(1-\sqrt{3}\right)=\left(1-\sqrt{3}\right)-\ell n\left(\frac{2-\left(4-2\sqrt{3}\right)}{2}\right)$$

$$=\left(1-\sqrt{3}\right)-\ell n\left(\sqrt{3}-1\right)$$

10. The Boolean expression $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to :

(1) ~q (2) q (3) p (4) ~p

Official Ans. by NTA (4)

Sol. $(p \to q) \wedge (q \to \sim p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee \sim p)$ $\{p \to q \equiv \sim p \vee q\}$

$\equiv (\sim p \vee q) \wedge (\sim p \vee \sim q)$ {commutative property}

$\equiv \sim p \vee (q \wedge \sim q)$ {distributive property}

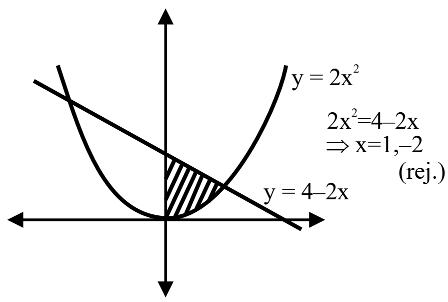
$\equiv \sim p$

11. The area (in sq. units) of the region, given by the set $\{(x, y) \in \mathbf{R}\times\mathbf{R} \mid x \geq 0, 2x^2 \leq y \leq 4-2x\}$ is :

(1) $\frac{8}{3}$ (2) $\frac{17}{3}$ (3) $\frac{13}{3}$ (4) $\frac{7}{3}$

Official Ans. by NTA (4)

Sol.



Required area $=\int_0^1\left(4-2x-2x^2\right)dx=4x-x^2-\frac{2x^3}{3}\Big|_0^1$

$$=4-1-\frac{2}{3}=\frac{7}{3}$$

12. The sum of all values of x in [0, 2π], for which sinx + sin2x + sin3x + sin4x = 0, is equal to :

(1) 8π (2) 11 π (3) 12 π (4) 9 π

Official Ans. by NTA (4)

Sol. (sinx + sin4x) + (sin2x + sin3x) = 0

$$\Rightarrow 2\sin\frac{5x}{2}\left\{\cos\frac{3x}{2}+\cos\frac{x}{2}\right\}=0$$

$$\Rightarrow 2\sin\frac{5x}{2}\left\{2\cos x\cos\frac{x}{2}\right\}=0$$

$$2\sin\frac{5x}{2}=0\Rightarrow\frac{5X}{2}=0,\pi,2\pi,3\pi,4\pi,5\pi$$

$$\Rightarrow x=0,\frac{2\pi}{5},\frac{4\pi}{5},\frac{6\pi}{5},\frac{8\pi}{5},2\pi$$

$$\cos\frac{x}{2}=0\Rightarrow\frac{x}{2}=\frac{\pi}{2}\Rightarrow x=\pi$$

$$\cos x=0\Rightarrow x=\frac{\pi}{2},\frac{3\pi}{2}$$

So sum = 6π + π + 2π = 9π

13. Let g : **N** → **N** be defined as

g(3n + 1) = 3n + 2,

g(3n + 2) = 3n + 3,

g(3n + 3) = 3n + 1, for all n ≥ 0.

Then which of the following statements is true ?

(1) There exists an onto function f : **N** → **N** such that fog = f

(2) There exists a one–one function f: **N** → **N** such that fog = f

(3) gogog = g

(4) There exists a function f : **N** → **N** such that gof = f

Official Ans. by NTA (1)

Sol. g : N → N g(3n + 1) = 3n + 2

g(3n + 2) = 3n + 3

g(3n + 3) = 3n + 1

$$g(x)=\begin{cases}x+1 & x=3k+1\\ x+1 & x=3k+2\\ x-2 & x=3k+3\end{cases}$$

$$g(g(x))=\begin{cases}x+2 & x=3k+1\\ x-1 & x=3k+2\\ x-1 & x=3k+3\end{cases}$$

$$g(g(g(x)))=\begin{cases}x & x=3k+1\\ x & x=3k+2\\ x & x=3k+3\end{cases}$$

If f : N → N, f is a one-one function such that f(g(x)) = f(x) ⇒ g(x) = x, which is not the case

If f f : N → N f is an onto function such that f(g(x)) = f(x),

one possibility is

$$f(x)=\begin{cases}n & x=3n+1\\ n & x=3n+2 \quad n\in N_0\\ n & x=3n+3\end{cases}$$

Here f(x) is onto, also f(g(x)) = f(x) ∀ x ∈ N

14. Let f : [0, ∞) → [0, ∞) be defined as

$$f(x)=\int_0^x[y]dy$$

where [x] is the greatest integer less than or equal to x. Which of the following is true?

(1) f is continuous at every point in [0, ∞) and differentiable except at the integer points.

(2) f is both continuous and differentiable except at the integer points in [0, ∞).

(3) f is continuous everywhere except at the integer points in [0, ∞).

(4) f is differentiable at every point in [0, ∞).

Official Ans. by NTA (1)

Sol. $f:[0,\infty)\to[0,\infty),\ f(x)=\int_0^x[y]\,dy$

Let x = n + f, f ∈ (0, 1)

So $f(x)=0+1+2+\ldots+(n-1)+\int_n^{n+f}n\,dy$

$$f(x)=\frac{n(n-1)}{2}+nf$$

$$=\frac{[x]([x]-1)}{2}+[x]\{x\}$$

Note $\lim_{x\to n^+}f(x)=\frac{n(n-1)}{2}$,

$$\lim_{x\to n^-}f(x)=\frac{(n-1)(n-2)}{2}+(n-1)=\frac{n(n-1)}{2}$$

$$f(x)=\frac{n(n-1)}{2}\quad(n\in N_0)$$

so f(x) is cont. ∀ x ≥ 0 and diff. except at integer points

15. The values of a and b, for which the system of equations

$$2x + 3y + 6z = 8$$
$$x + 2y + az = 5$$
$$3x + 5y + 9z = b$$

has no solution, are :

(1) $a = 3, b \neq 13$ (2) $a \neq 3, b \neq 13$

(3) $a \neq 3, b = 3$ (4) $a = 3, b = 13$

Official Ans. by NTA (1)

Sol. $D = \begin{vmatrix} 2 & 3 & 6 \\ 1 & 2 & a \\ 3 & 5 & 9 \end{vmatrix} = 3 - a$

$D = \begin{vmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \\ 3 & 5 & b \end{vmatrix} = b - 13$

If $a = 3, b \neq 13$, no solution.

16. Let 9 distinct balls be distributed among 4 boxes, B_1, B_2, B_3 and B_4. If the probability than B_3 contains exactly 3 balls is $k\left(\frac{3}{4}\right)^9$ then k lies in the set :

(1) $\{x \in \mathbf{R} : |x - 3| < 1\}$ (2) $\{x \in \mathbf{R} : |x - 2| \leq 1\}$

(3) $\{x \in \mathbf{R} : |x - 1| < 1\}$ (4) $\{x \in \mathbf{R} : |x - 5| \leq 1\}$

Official Ans. by NTA (1)

Sol. required probability $= \frac{{}^9C_3 . 3^6}{4^9}$

$= \frac{{}^9C_3}{27} . \left(\frac{3}{4}\right)^9$

$= \frac{28}{9} . \left(\frac{3}{4}\right)^9 \Rightarrow k = \frac{28}{9}$

Which satisfies $|x - 3| < 1$

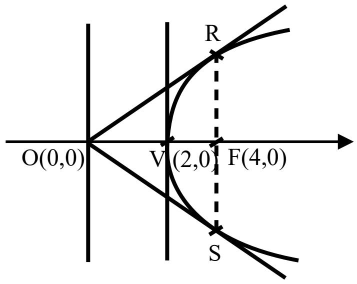
17. Let a parabola P be such that its vertex and focus lie on the positive x–axis at a distance 2 and 4 units from the origin, respectively. If tangents are drawn from O(0, 0) to the parabola P which meet P at S and R, then the area (in sq. units) of ΔSOR is equal to :

(1) $16\sqrt{2}$ (2) 16

(3) 32 (4) $8\sqrt{2}$

Official Ans. by NTA (2)

Sol.



Clearly RS is latus-rectum

$\because$ VF = 2 = a

$\therefore$ RS = 4a = 8

Now OF = 2a = 4

$\Rightarrow$ Area of triangle ORS = 16

18. The number of real roots of the equation $e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^x + 1 = 0$ is :

(1) 2 (2) 4 (3) 6 (4) 1

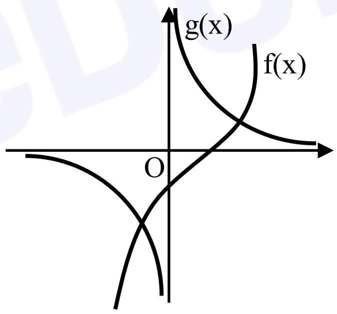
Official Ans. by NTA (1)

Sol. $e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^x + 1 = 0$

$\Rightarrow \left(e^{3x} - 1\right)^2 - e^x\left(e^{3x} - 1\right) = 12e^{2x}$

$\left(e^{3x} - 1\right)^2\left(e^x - e^{-x} - e^{-2x}\right) = 12$

$\Rightarrow \underbrace{e^x - e^{-x} - e^{-2x}}_{\text{increasing (let } f(x))} = \underbrace{\frac{12}{e^{3x} - 1}}_{\text{decreasing (let } g(x))}$



$\Rightarrow$ No. of real roots = 2

19. Let an ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a^2 > b^2$, passes through $\left(\sqrt{\frac{3}{2}}, 1\right)$ and has eccentricity $\frac{1}{\sqrt{3}}$. If a circle, centered at focus $F(\alpha, 0)$, $\alpha > 0$, of E and radius $\frac{2}{\sqrt{3}}$, intersects E at two points P and Q, then PQ^2 is equal to :

(1) $\frac{8}{3}$ (2) $\frac{4}{3}$ (3) $\frac{16}{3}$ (4) 3

Official Ans. by NTA (3)

Sol. $\dfrac{3}{2a^2}+\dfrac{1}{b^2}=1$ and $1-\dfrac{b^2}{a^2}=\dfrac{1}{3}$

$\Rightarrow a^2 = 3\ b^2 = 3$

$\Rightarrow \dfrac{x^2}{3}+\dfrac{y^2}{2}=1$ .....(i)

Its focus is (1,0)

Now, eqn of circle is

$(x-1)^2+y^2=\dfrac{4}{3}$ .....(ii)

Solving (i) and (ii) we get

$y=\pm\dfrac{2}{\sqrt{3}}, x=1$

$\Rightarrow PQ^2=\left(\dfrac{4}{\sqrt{3}}\right)^2=\dfrac{16}{3}$

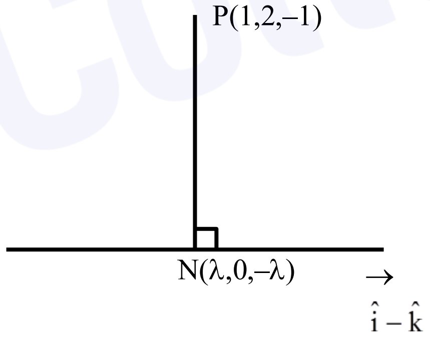
20. Let the foot of perpendicular from a point P(1, 2, –1) to the straight line $L:\dfrac{x}{1}=\dfrac{y}{0}=\dfrac{z}{-1}$ be N. Let a line be drawn from P parallel to the plane $x + y + 2z = 0$ which meets L at point Q. If α is the acute angle between the lines PN and PQ, then $\cos\alpha$ is equal to __________.

(1) $\dfrac{1}{\sqrt{5}}$ (2) $\dfrac{\sqrt{3}}{2}$

(3) $\dfrac{1}{\sqrt{3}}$ (4) $\dfrac{1}{2\sqrt{3}}$

Official Ans. by NTA (3)

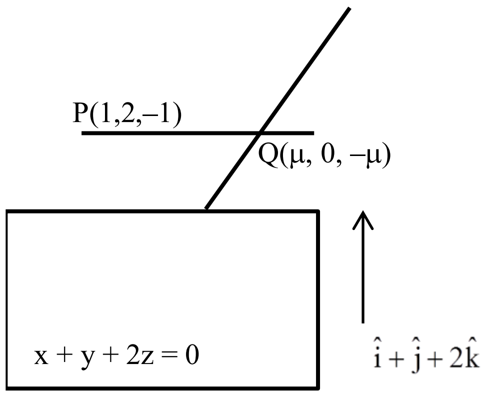
Sol.



$\overrightarrow{PN}.(\hat{i}-\hat{k})=0$

$\Rightarrow$ N(1, 0, –1)

Now,



$\overrightarrow{PQ}.\left(\hat{i}+\hat{j}+2\hat{k}\right)=0$

$\Rightarrow \mu = -1$

$\Rightarrow$ Q (–1, 0, 1)

$\overrightarrow{PN}=2\hat{j}$ and $\overrightarrow{PQ}=2\hat{i}+2\hat{j}-2\hat{k}$

$\Rightarrow \cos\alpha=\dfrac{1}{\sqrt{3}}$

SECTION-B

1. Let $y = y(x)$ be solution of the following differential equation

$e^y\dfrac{dy}{dx}-2e^y\ \sin x+\sin x\cos^2 x=0,\ y\left(\dfrac{\pi}{2}\right)=0$

If $y(0) = \log_e(\alpha + \beta e^{-2})$, then $4(\alpha + \beta)$ is equal to __________.

Official Ans. by NTA (4)

Sol. Let $e^y = t$

$\Rightarrow \dfrac{dt}{dx}-(2\sin x)t=-\sin x\cos^2 x$

I.F. $= e^{2\cos x}$

$\Rightarrow t\cdot e^{2\cos x}=\int e^{2\cos x}.(-\sin x\cos^2 x)dx$

$\Rightarrow e^y\cdot e^{2\cos x}=\int e^{2z}\cdot z^2dz, z=e^{2\cos x}$

$\Rightarrow e^y\cdot e^{2\cos x}=\dfrac{1}{2}.\cos^2 x\cdot e^{2\cos x}-\dfrac{1}{2}\cos x\cdot e^{2\cos x}+\dfrac{e^{2\cos x}}{4}+C$

at $x=\dfrac{\pi}{2}, y=0\Rightarrow C=\dfrac{3}{4}$

$\Rightarrow e^y=\dfrac{1}{2}\cos^2 x-\dfrac{1}{2}\cos x+\dfrac{1}{4}+\dfrac{3}{4}\cdot e^{-2\cos x}$

$\Rightarrow y=\log\left[\dfrac{\cos^2 x}{2}-\dfrac{\cos x}{2}+\dfrac{1}{4}+\dfrac{3}{4}e^{-2\cos x}\right]$

Put x = 0

$\Rightarrow y = \log\left[\frac{1}{4} + \frac{3}{4}e^{-2}\right] \Rightarrow \alpha = \frac{1}{4}, \beta = \frac{3}{4}$

2. If the value of

$$\left(1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \text{......upto } \infty\right)^{\log_{(0.25)}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\text{.......upto } \infty\right)}$$

is l, then l^2 is equal to ________.

Official Ans. by NTA (3)

Sol. $\ell = \left(\underbrace{1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \ldots}_{S}\right)^{\log_{0.25}\left(\frac{1}{3}+\frac{1}{3^2}+\ldots\right)}$

$S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \ldots.$

$\frac{S}{3} = \quad \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \ldots.$

$- \quad - \quad - \quad -$

$\frac{2S}{3} = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \ldots.$

$\frac{2S}{3} = \frac{4}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \ldots.$

$S = \frac{3}{2}\left(\frac{4/3}{1-1/3}\right) = 3$

Now $\ell = (3)^{\log_{0.25}\left(\frac{1/3}{1-1/3}\right)}$

$\ell = 3^{\log_{((1/4))}\left(\frac{1}{2}\right)} = 3^{1/2} = \sqrt{3}$

$\Rightarrow \ell^2 = 3$

3. Consider the following frequency distribution :

class :	10–20	20–30	30–40	40–50	50–60
Frequency :	α	110	54	30	β

If the sum of all frequencies is 584 and median is 45, then $|\alpha - \beta|$ is equal to _________.

Official Ans. by NTA (164)

Sol. $\because$ Sum of frequencies = 584

$\Rightarrow \alpha + \beta = 390$

Now, Median is at $\frac{584}{2} = 292^{th}$

$\because$ Median = 45 (lies in class 40 – 50)

$\Rightarrow \alpha + 110 + 54 + 15 = 292$

$\Rightarrow \alpha = 113, \beta = 277$

$\Rightarrow |\alpha - \beta| = 164$

4. Let $\vec{p} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{q} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. If a vector $\vec{r} = (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$ is perpendicular to each of the vectors $(\vec{p} + \vec{q})$ and $(\vec{p} - \vec{q})$, and $|\vec{r}| = \sqrt{3}$, then $|\alpha| + |\beta| + |\gamma|$ is equal to _______.

Official Ans. by NTA (3)

Sol. $\vec{p} = 2\hat{i} + 3\hat{j} + \hat{k}$ (Given)

$\vec{q} = \hat{i} + 2\hat{j} + \hat{k}$

Now $(\vec{p} + \vec{q}) \times (\vec{p} - \vec{q}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 2 \\ 1 & 1 & 0 \end{vmatrix}$

$= -2\hat{i} - 2\hat{j} - 2\hat{k}$

$\Rightarrow \vec{r} = \pm\sqrt{3}\frac{((\vec{p}+\vec{q})\times(\vec{p}-\vec{q}))}{|(\vec{p}+\vec{q})\times(\vec{p}-\vec{q})|} = \pm\frac{\sqrt{3}(-2\hat{i}-2\hat{j}-2\hat{k})}{\sqrt{2^2+2^2+2^2}}$

$\vec{r} = \pm(-\hat{i} - \hat{j} - \hat{k})$

According to question

$\vec{r} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

So $|\alpha| = 1, |\beta| = 1, |\gamma| = 1$

$\Rightarrow |\alpha| + |\beta| + |\gamma| = 3$

5. The ratio of the coefficient of the middle term in the expansion of $(1 + x)^{20}$ and the sum of the coefficients of two middle terms in expansion of $(1 + x)^{19}$ is _________.

Official Ans. by NTA (1)

Sol. Coeff. of middle term in $(1 + x)^{20} = {}^{20}C_{10}$

& Sum of Coeff. of two middle terms in

$(1 + x)^{19} = {}^{19}C_9 + {}^{19}C_{10}$

So required ratio $= \frac{{}^{20}C_{10}}{{}^{19}C_9 + {}^{19}C_{10}} = \frac{{}^{20}C_{10}}{{}^{20}C_{10}} = 1$

6. Let $M = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \{\pm3, \pm2, \pm1, 0\}\right\}$.

Define $f : M \rightarrow \mathbf{Z}$, as $f(A) = \det(A)$, for all $A \in M$, where $\mathbf{Z}$ is set of all integers. Then the number of $A \in M$ such that $f(A) = 15$ is equal to ________.

Official Ans. by NTA (16)

Sol. $|A| = ad - bc = 15$

where a,b,c,d $\in \{\pm 3, \pm 2, \pm 1, 0\}$

Case I ad = 9 & bc = – 6

For ad possible pairs are (3,3), (–3,–3)

For bc possible pairs are (3,–2), (–3,2), (–2,3),(2,–3)

So total matrix = 2 × 4 = 8

Case II ad = 6 & bc = –9

Similarly total matrix = 2 × 4 = 8

$\Rightarrow$ Total such matrices are = 16

7. There are 5 students in class 10, 6 students in class 11 and 8 students in class 12. If the number of ways, in which 10 students can be selected from them so as to include at least 2 students from each class and at most 5 students from the total 11 students of class 10 and 11 is 100 k, then k is equal to _______.

Official Ans. by NTA (238)

Sol.

Class	10^{th}	11^{th}	12^{th}	
Total student	5	6	8	
	2	3	5	$\Rightarrow {}^5C_2 \times {}^6C_3 \times {}^8C_5$
Number of selection	2	2	6	$\Rightarrow {}^5C_2 \times {}^6C_2 \times {}^8C_6$
	3	2	5	$\Rightarrow {}^5C_3 \times {}^6C_2 \times {}^8C_5$

$\Rightarrow$ Total number of ways = 23800

According to question

100 K = 23800

$\Rightarrow$ K = 238

8. If α, β are roots of the equation $x^2 + 5(\sqrt{2})x + 10 = 0$, $\alpha > \beta$ and $P_n = \alpha^n - \beta^n$ for each positive integer n, then the value of $\left(\frac{P_{17}P_{20} + 5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}P_{18}^2}\right)$ is equal to _________.

Official Ans. by NTA (1)

Sol. $x^2+5\sqrt{2}\,x+10=0$

& $p_n = \alpha^n - \beta^n$ (Given)

Now $\frac{P_{17}P_{20} + 5\sqrt{2}\,P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}P_{18}^2} = \frac{P_{17}(P_{20} + 5\sqrt{2}\,P_{19})}{P_{18}(P_{19} + 5\sqrt{2}P_{18})}$

$$\frac{P_{17}(\alpha^{20} - \beta^{20} + 5\sqrt{2}(\alpha^{19} - \beta^{19}))}{P_{18}(\alpha^{19} - \beta^{19} + 5\sqrt{2}(\alpha^{18} - \beta^{18}))}$$

$$\frac{P_{17}(\alpha^{19}(\alpha + 5\sqrt{2}) - \beta^{19}(\beta + 5\sqrt{2}))}{P_{18}(\alpha^{18}(\alpha + 5\sqrt{2}) - \beta^{18}(\beta + 5\sqrt{2}))}$$

Since $\alpha + 5\sqrt{2} = -10/\alpha$ ....(1)

& $\beta + 5\sqrt{2} = -10/\beta$ ....(2)

Now put there values in above expression

$$= -\frac{10\,P_{17}\,P_{18}}{-10\,P_{18}\,P_{17}} = 1$$

9. The term independent of 'x' in the expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$, where $x \neq 0, 1$ is equal to ________.

Official Ans. by NTA (210)

Sol. $\left(\left(x^{1/3}+1\right) - \left(\frac{x^{1/2}+1}{x^{1/2}}\right)\right)^{10}$

$$= \left(x^{1/3} - \frac{1}{x^{1/2}}\right)^{10}$$

Now General Term

$$T_{r+1} = {}^{10}C_r \left(x^{1/3}\right)^{10-r} \cdot \left(-\frac{1}{x^{1/2}}\right)^r$$

For independent term

$$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow r = 4$$

$\Rightarrow T_5 = {}^{10}C_4 = 210$

10. Let

$$S = \left\{ n \in \mathbf{N} \middle| \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \forall a,b,c,d \in \mathbf{R} \right\},$$

where $i = \sqrt{-1}$. Then the number of 2–digit numbers in the set S is _______.

Official Ans. by NTA (11)

Sol. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ & $A = \begin{pmatrix} o & i \\ 1 & 0 \end{pmatrix}^n$

$\Rightarrow AX = IX$

$\Rightarrow A = I$

$$\Rightarrow \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}^n = I$$

$$\Rightarrow A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ n is multiple of 8

So number of 2 digit numbers in the set

S = 11 (16, 24, 32, ...,96)