CollegeDekho

FINAL JEE–MAIN EXAMINATION – FEBRUARY, 2021

(Held On Friday 26th February, 2021) TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. If vectors $\vec{a}_1 = x\hat{i} - \hat{j} + \hat{k}$ and $\vec{a}_2 = \hat{i} + y\hat{j} + z\hat{k}$ are collinear, then a possible unit vector parallel to the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is

(1) $\frac{1}{\sqrt{2}}\left(-\hat{j}+\hat{k}\right)$ (2) $\frac{1}{\sqrt{2}}\left(\hat{i}-\hat{j}\right)$

(3) $\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}-\hat{k}\right)$ (4) $\frac{1}{\sqrt{3}}\left(\hat{i}-\hat{j}+\hat{k}\right)$

Official Ans. by NTA (4)

Sol. $\vec{a}_1$ and $\vec{a}_2$ are collinear

so $\frac{x}{1} = \frac{-1}{y} = \frac{1}{z}$

unit vector in direction of

$$x\hat{i} + y\hat{j} + z\hat{k} = \pm\frac{1}{\sqrt{3}}\left(\hat{i}-\hat{j}+\hat{k}\right)$$

2. Let A = {1, 2, 3, ..., 10} and f : A → A be defined as $f(k) = \begin{cases} k+1 & \text{if k is odd} \\ k & \text{if k is even} \end{cases}$

Then the number of possible functions g : A → A such that g∘f = f is

(1) 10^5 (2) $^{10}C_5$ (3) 5^5 (4) 5!

Official Ans. by NTA (1)

Sol. $f(x) = \begin{cases} x+1, & \text{if x is odd} \\ x, & \text{if x is even} \end{cases}$

∵ g : A → A such that $g(f(x)) = f(x)$

⇒ If x is even then g(x) = x ...(1)

If x is odd then g(x + 1) = x + 1 ...(2)

from (1) and (2) we can say that

g(x) = x if x is even

⇒ If x is odd then g(x) can take any value in

3. Let f : R → R be defined as

$$f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), & \text{if } x < -1 \\ |ax^2 + x + b|, & \text{if } -1 \le x \le 1 \\ \sin(\pi x), & \text{if } x > 1 \end{cases}$$

If $f(x)$ is continuous on R, then a + b equals:

(1) –3 (2) –1 (3) 3 (4) 1

Official Ans. by NTA (2)

Sol. $f(x)$ is continuous on R

⇒ $f(1^-) = f(1) = f(1^+)$

$|a + 1 + b| = \lim_{x\to 1}\sin(\pi x)$

$|a + 1 + b| = 0 \Rightarrow a + b = -1$ ...(1)

⇒ Also $f(-1^-) = f(-1) = f(-1^+)$

$$\lim_{x\to -1} 2\sin\left(\frac{-\pi x}{2}\right) = |a - 1 + b|$$

$|a - 1 + b| = 2$

Either a – 1 + b = 2 or a – 1 + b = –2

a + b = 3 ...(2) or a + b = –1 ...(3)

from (1) and (2) ⇒ a + b = 3 = – 1(reject)

from (1) and (3) ⇒ a + b = –1

4. For $x > 0$, if $f(x) = \int_1^x \frac{\log_e t}{(1+t)}dt$, then $f(e) + f\left(\frac{1}{e}\right)$ is equal to

(1) 1 (2) –1 (3) $\frac{1}{2}$ (4) 0

Official Ans. by NTA (3)

Sol. $f(x) = \int_1^x \frac{\log_e t}{(1+t)}dt$

$f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ell nt}{(1+t)}dt$, let $t = \frac{1}{}$

$$= +\int_1^x \frac{\ell ny}{1+y}.\frac{y}{y^2}dy$$

$$= \int_1^x \frac{\ell ny}{y(1+y)}dy$$

hence

$$f(x)+f\left(\frac{1}{x}\right) = \int_1^x \frac{(1+t)\ell nt}{t(1+t)}dt = \int_1^x \frac{\ell nt}{t}dt$$

$$= \frac{1}{2}\ell n^2(x)$$

so $f(e)+f\left(\frac{1}{e}\right) = \frac{1}{2}$ ..(3)

5. A natural number has prime factorization given by $n = 2^x3^y5^z$, where y and z are such that $y + z = 5$ and $y^{-1} + z^{-1} = \frac{5}{6}$, $y > z$. Then the number of odd divisors of n, including 1, is :

(1) 11 (2) 6 (3) 6x (4) 12

Official Ans. by NTA (4)

Sol. Ans. (4)

Sol. $y + z = 5$

$\frac{1}{y}+\frac{1}{z}=\frac{5}{6}$ $y > z$

$\Rightarrow y = 3, z = 2$

$\Rightarrow n = 2^x.3^3.5^2 = (2.2.2 ...) (3.3.3) (5.5)$

Number of odd divisors $= 4 \times 3 = 12$

6. Let $f(x) = \sin^{-1}x$ and $g(x)=\frac{x^2-x-2}{2x^2-x-6}$. If $g(2)=\lim\limits_{x\to 2} g(x)$, then the domain of the function fog is :

(1) $(-\infty,-2]\cup\left[-\frac{3}{2},\infty\right)$

(2) $(-\infty,-2]\cup[-1,\infty)$

(3) $(-\infty,-2]\cup\left[-\frac{4}{3},\infty\right)$

(4) $(-\infty,-1]\cup[2,\infty)$

Sol. Domain of $fog(x) = \sin^{-1}(g(x))$

$\Rightarrow |g(x)| \le 1$, $g(2) = \frac{3}{7}$

$$\left|\frac{x^2-x-2}{2x^2-x-6}\right|\le 1$$

$$\left|\frac{(x+1)(x-2)}{(2x+3)(x-2)}\right|\le 1$$

$$\frac{x+1}{2x+3}\le 1 \text{ and } \frac{x+1}{2x+3}\ge -1$$

$$\frac{x+1-2x-3}{2x+3}\le 0 \text{ and } \frac{x+1+2x+3}{2x+3}\ge 0$$

$$\frac{x+2}{2x+3}\ge 0 \text{ and } \frac{3x+4}{2x+3}\ge 0$$

$$x\in(-\infty,-2]\cup\left[-\frac{4}{3},\infty\right)$$

7. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is :

(1) An isosceles triangle with base equal to 2r.

(2) An equilateral triangle of height $\frac{2r}{3}$.

(3) An equilateral triangle having each of its side of length $\sqrt{3}r$.

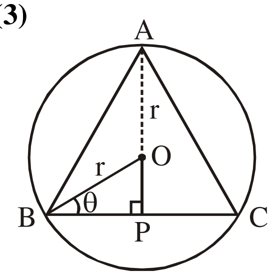
(4) A right angle triangle having two of its sides of length 2r and r.

Official Ans. by NTA (3)

Sol. $h = r\sin\theta + r$

base $= BC = 2r\cos\theta$

$\theta\in\left[0,\frac{\pi}{2}\right)$



Area of $\Delta ABC = \frac{1}{2}(BC).h$

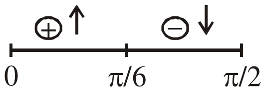
$$\Delta = \frac{1}{2}(2r\cos\theta).(r\sin\theta + r)$$

$= r^2 (\cos\theta).(1 + \sin\theta)$

$$\frac{d\Delta}{d\theta} = r^2\left[\cos^2\theta - \sin\theta - \sin^2\theta\right]$$

$= r^2\underbrace{[1+\sin\theta]}_{\text{positive}}[1-2\sin\theta]=0$

$\Rightarrow \theta = \frac{\pi}{6}$



$\Rightarrow \Delta$ is maximum where $\theta = \frac{\pi}{6}$

$\Delta_{max.} = \frac{3\sqrt{3}}{4}r^2$ = area of equilateral Δ with $BC = \sqrt{3}r$.

8. Let L be a line obtained from the intersection of two planes x + 2y + z = 6 and y + 2z = 4. If point P(α, β, γ) is the foot of perpendicular from (3, 2, 1) on L, then the value of 21(α + β + γ) equals :

(1) 142 (2) 68 (3) 136 (4) 102

Official Ans. by NTA (4)

Sol. $x + 2y + z = 6$

$(y + 2z = 4) \times 2$

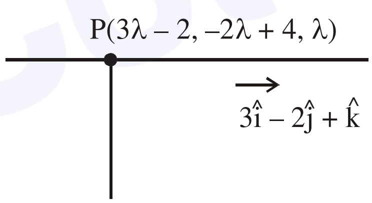
$x - 3z = -2 \Rightarrow x = 3z - 2 \Rightarrow y = 4 - 2z$

$\frac{x+2}{3} = z \qquad \frac{y-4}{-2} = z$

$\Rightarrow$ line of intersection of two planes is

$\frac{x+2}{3} = \frac{y-4}{-2} = z = \lambda$ (Let)

$\because$ AP $\perp^{ar}$ to line



$\therefore \overline{AP}.\left(3\hat{i}-2\hat{j}+\hat{k}\right)=0$

$(3\lambda - 5) . 3 + (-2\lambda + 2)(-2) + (\lambda - 1) . 1 = 0$

$9\lambda - 15 + 4\lambda - 4 + \lambda - 1 = 0$

$14\lambda = 20$

$\lambda = \frac{10}{} \Rightarrow P\left(\frac{16}{}, \frac{8}{}, \frac{10}{}\right)$

$\Rightarrow \alpha + \beta + \gamma = \frac{16+8+10}{7} = \frac{34}{7}$

$21(\alpha + \beta + \gamma) = 102$

9. Let $F_1(A,B,C) = (A\wedge\sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$ and $F_2(A, B) = (A \vee B) \vee (B \rightarrow \sim A)$ be two logical expressions. Then :

(1) F_1 and F_2 both are tautologies

(2) F_1 is a tautology but F_2 is not a tautology

(3) F_1 is not tautology but F_2 is a tautology

(4) Both F_1 and F_2 are not tautologies

Official Ans. by NTA (3)

Sol. $F_1 : (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$

$F_2 : (A \vee B) \vee (B \rightarrow \sim A)$

$F_1 : \{(A \wedge \sim B) \vee \sim A\} \vee [(A \vee B) \wedge \sim C]$

$: \{(A \vee \sim A) \wedge (\sim A \vee \sim B)\} \vee [(A \vee B) \wedge \sim C]$

$: \{t \wedge (\sim A \vee \sim B)\} \vee [(A \vee B) \wedge \sim C]$

$: (\sim A \vee \sim B) \vee [(A \vee B) \wedge \sim C]$

$: \underbrace{[(\sim A \vee \sim B) \vee (A \vee B)]}_{t} \wedge [(\sim A \vee \sim B) \wedge \sim C]$

$F_1 : (\sim A \vee \sim B) \wedge \sim C \neq t$ (tautology)

$F_2 : (A \vee B) \vee (\sim B \vee \sim A) = t$ (tautology)

10. Let slope of the tangent line to a curve at any point P(x, y) be given by $\frac{xy^2+y}{x}$. If the curve intersects the line x + 2y = 4 at x = –2, then the value of y, for which the point (3, y) lies on the curve, is :

(1) $\frac{18}{35}$ (2) $-\frac{4}{3}$ (3) $-\frac{18}{19}$ (4) $-\frac{18}{11}$

Official Ans. by NTA (3)

Sol. $\frac{dy}{dx} = \frac{xy^2+y}{x}$

$\frac{xdy - ydx}{y^2} = xdx$

$-d\left(\frac{x}{y}\right) = xdx$

$-\frac{x}{} = \frac{x^2}{} + c$

$\because$ curve intersects the line x + 2y = 4 at x = –2 $\Rightarrow$ point of intersection is (–2, 3)

$\therefore$ curve passes through (–2, 3)

$$\Rightarrow \frac{2}{3} = 2 + c \Rightarrow c = -\frac{4}{3}$$

$$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$$

Now put (3, y)

$$\Rightarrow \frac{-3}{y} = \frac{19}{6}$$

$$\Rightarrow y = \frac{-18}{19}$$

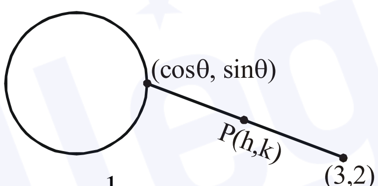
11. If the locus of the mid-point of the line segment from the point (3, 2) to a point on the circle, $x^2 + y^2 = 1$ is a circle of radius r, then r is equal to :

(1) 1 (2) $\frac{1}{2}$ (3) $\frac{1}{3}$ (4) $\frac{1}{4}$

Official Ans. by NTA (2)

Sol. $h = \frac{\cos\theta + 3}{2}$

$k = \frac{\sin\theta + 2}{2}$



$$\Rightarrow \left(h - \frac{3}{2}\right)^2 + (k-1)^2 = \frac{1}{4}$$

$$\Rightarrow r = \frac{1}{2}$$

12. Consider the following system of equations :

x + 2y – 3z = a

2x + 6y – 11z = b

x – 2y + 7z = c,

where a, b and c are real constants. Then the system of equations :

(1) has a unique solution when 5a = 2b + c

(2) has infinite number of solutions when 5a = 2b + c

(3) has no solution for all a, b and c

(4) has a unique solution for all a, b and c

Sol. P_1 : x + 2y – 3z = a

P_2 : 2x + 6y – 11z = b

P_3 : x – 2y + 7z = c

Clearly

$5P_1 = 2P_2 + P_3$ if 5a = 2b + c

$\Rightarrow$ All the planes sharing a line of intersection

$\Rightarrow$ infinite solutions

13. If 0 < a, b < 1, and $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$, then the value of

$$(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + \ldots$$

is :

(1) $\log_e 2$ (2) $e^2 - 1$

(3) e (4) $\log_e\left(\frac{e}{2}\right)$

Official Ans. by NTA (1)

Sol. $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$ 0 < a, b < 1

$$\Rightarrow \frac{a+b}{1-ab} = 1$$

a + b = 1 – ab

(a + 1)(b + 1) = 2

Now $\left[a - \frac{a^2}{2} + \frac{a^3}{3} + \ldots\right] + \left[b - \frac{b^2}{2} + \frac{b^3}{3} + \ldots\right]$

$= \log_e(1 + a) + \log_e(1 + b)$

($\because$ expansion of $\log_e(1 + x)$)

$= \log_e[(1 + a)(1 + b)]$

$= \log_e 2$

14. The sum of the series $\sum_{n=1}^{\infty} \frac{n^2 + 6n + 10}{(2n+1)!}$ is equal to :

(1) $\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$

(2) $\frac{41}{8}e - \frac{19}{8}e^{-1} - 10$

(3) $\frac{41}{8}e + \frac{19}{8}e^{-1} + 10$

(4) $-\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$

Official Ans. by NTA (2)

Sol. $T_n = \frac{n^2+6n+10}{(2n+1)!} = \frac{4n^2+24n+40}{4.(2n+1)!}$

$= \frac{(2n+1)^2+20n+39}{4.(2n+1)!}$

$= \frac{(2n+1)^2+(2n+1).10+29}{4(2n+1)!}$

$= \frac{1}{4}\left[\frac{(2n+1)^2}{(2n+1)(2n)!} + \frac{(2n+1)10}{(2n+1)(2n)!} + \frac{29}{(2n+1)!}\right]$

$= \frac{1}{4}\left[\frac{2n+1}{(2n)!} + \frac{10}{(2n)!} + \frac{29}{(2n+1)!}\right]$

$= \frac{1}{4}\left[\frac{1}{(2n-1)!} + \frac{11}{(2n)!} + \frac{29}{(2n+1)!}\right]$

$S_1 = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + ... = \frac{e - \frac{1}{e}}{2}$

$S_2 = 11\left[\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...\right] = 11\left[\frac{e + \frac{1}{e} - 2}{2}\right]$

$S_3 = 29\left[\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + ...\right] = 29\left[\frac{e - \frac{1}{e} - 2}{2}\right]$

Now, $S = \frac{1}{4}[S_1 + S_2 + S_3]$

$= \frac{1}{4}\left[\frac{e}{2} - \frac{1}{2e} + \frac{11e}{2} + \frac{11}{2e} + \frac{29e}{2} - \frac{29}{2e} - 4\right]$

$= \frac{41e}{8} - \frac{19}{8e} - 10$

15. Let $f(x)$ be a differentiable function at x = a with $f'(a) = 2$ and $f(a) = 4$. Then $\lim_{x \to a} \frac{xf(a) - af(x)}{x - a}$ equals :

(1) 2a + 4 (2) 4 – 2a
(3) 2a – 4 (4) a + 4

Official Ans. by NTA (2)

Sol. $f'(a) = 2$, $f(a) = 4$

$\lim_{x \to a} \frac{xf(a) - af(x)}{x - a}$

$\Rightarrow \lim_{x \to a} \frac{f(a) - af'(x)}{1}$ (Lopitals rule)

$= f(a) - af'(a)$

= 4 – 2a

16. Let A(1, 4) and B(1, –5) be two points. Let P be a point on the circle $(x - 1)^2 + (y - 1)^2 = 1$ such that $(PA)^2 + (PB)^2$ have maximum value, then the points P, A and B lie on :

(1) a straight line (2) a hyperbola
(3) an ellipse (4) a parabola

Official Ans. by NTA (1)

Sol. P be a point on $(x - 1)^2 + (y - 1)^2 = 1$

so P(1 + cosθ, 1 + sinθ)

A(1,4) B(1,–5)

$(PA)^2 + (PB)^2$

$= (\cos\theta)^2 + (\sin\theta - 3)^2 + (\text{cso}\theta)^2 + (\sin\theta + 6)^2$

= 47 + 6sinθ

is maximum if sinθ = 1

⇒ sinθ = 1, cosθ = 0

P(1,1) A(1,4) B(1,–5)

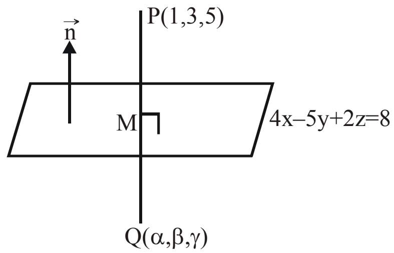
P,A,B are collinear points.

17. If the mirror image of the point (1, 3, 5) with respect to the plane 4x – 5y + 2z = 8 is (α,β, γ), then 5(α + β + γ) equals :

(1) 47 (2) 43 (3) 39 (4) 41

Official Ans. by NTA (1)

Sol.



Point Q is image of point P w.r.to plane, M is mid point of P and Q, lies in plane

$$M\left(\frac{1+\alpha}{2},\frac{3+\beta}{2},\frac{5+\gamma}{2}\right)$$

$4x - 5y + 2z = 8$

$$4\left(\frac{1+\alpha}{2}\right)-5\left(\frac{3+\beta}{2}\right)+2\left(\frac{5+\gamma}{2}\right)=8 \quad ..(1)$$

Also PQ perpendicualr to the plane

$\Rightarrow \overrightarrow{PQ} \parallel \vec{n}$

$$\frac{\alpha-1}{4}=\frac{\beta-3}{-5}=\frac{\gamma-5}{2}=k \text{ (let)}$$

$$\left.\begin{aligned}\alpha&=1+4k\\ \beta&=3-5k\\ \gamma&=5+2k\end{aligned}\right\} \quad ...(2)$$

use (2) in (1)

$$2(1+4k)-5\left(\frac{6-5k}{2}\right)+(10+2k)=8$$

$$k=\frac{2}{5}$$

from (2) $\alpha=\frac{13}{5},\beta=1,\gamma=\frac{29}{5}$

$5(\alpha + \beta + \gamma) = 13 + 5 + 29 = 47$

18. Let $f(x)=\int_0^x e^t f(t)dt+e^x$ be a differentiable function for all $x \in R$. Then $f(x)$ equals :

(1) $2e^{(e^x-1)}-1$ (2) $e^{e^x}-1$

(3) $2e^{e^x}-1$ (4) $e^{(e^x-1)}$

Official Ans. by NTA (1)

Sol. $f(x)=\int_0^x e^t f(t)\,dt+e^x \Rightarrow f(0) = 1$

differentiating with respect to x

$f'(x) = e^x f(x) + e^x$

$f'(x) = e^x(f(x) + 1)$

$$\int_0^x \frac{f'(x)}{f(x)+1}dx=\int_0^x e^x\,dx$$

$$\ell n(f(x)+1)\Big|_0^x=e^x\Big|_0^x$$

$\ell n(f(x) + 1) - \ell n(f(0) + 1) = e^x - 1$

$$\ell n\left(\frac{f(x)+1}{2}\right)=e^x-1 \qquad \{\text{as } f(0) = 1\}$$

$$f(x)=2e^{(e^x-1)}-1$$

19. Let A_1 be the area of the region bounded by the curves y = sinx, y = cosx and y-axis in the first quadrant. Also, let A_2 be the area of the region bounded by the curves y = sinx, y = cosx, x-axis and $x=\frac{\pi}{2}$ in the first quadrant.

Then,

(1) $A_1 : A_2 = 1:\sqrt{2}$ and $A_1 + A_2 = 1$

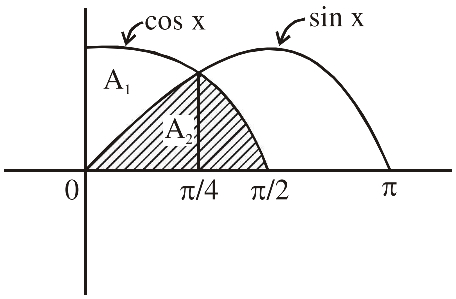
(2) $A_1 = A_2$ and $A_1 + A_2 = \sqrt{2}$

(3) $2A_1 = A_2$ and $A_1 + A_2 = 1+\sqrt{2}$

(4) $A_1 : A_2 = 1:2$ and $A_1 + A_2 = 1$

Official Ans. by NTA (1)

Sol.



$$A_1 = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$A_1 = (\sin x + \cos x)_0^{\pi/4} = \sqrt{2} - 1$$

$$A_2 = \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$$

$$= (-\cos x)_0^{\pi/4} + (\sin x)_{\pi/4}^{\pi/2}$$

$$A_2 = \sqrt{2}\left(\sqrt{2} - 1\right)$$

$A_1 : A_2 = 1 : \sqrt{2}$, $A_1 + A_2 = 1$

20. A seven digit number is formed using digits 3, 3, 4, 4, 4, 5, 5. The probability, that number so formed is divisible by 2, is :

(1) $\frac{6}{7}$ (2) $\frac{1}{7}$ (3) $\frac{3}{7}$ (4) $\frac{4}{7}$

Official Ans. by NTA (3)

Sol. Digits = 3, 3, 4, 4, 4, 5, 5

Total 7 digit numbers = $\frac{7!}{2!2!3!}$

Number of 7 digit number divisible by 2

$\Rightarrow$ last digit = 4

						4

3 , 3 , 4 , 4 , 5 , 5

Now 7 digit numbers which are divisible by 2

$= \frac{6!}{2!2!2!}$

Required probability $= \dfrac{\frac{6!}{2!2!2!}}{\frac{7!}{2!2!3!}} = \frac{3}{7}$

SECTION B

1. Let z be those complex numbers which satisfy $|z + 5| \le 4$ and $z(1+i) + \bar{z}(1-i) \ge -10, i = \sqrt{-1}$.

If the maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$, then the value of $(\alpha + \beta)$ is _______.

Official Ans. by NTA (48)

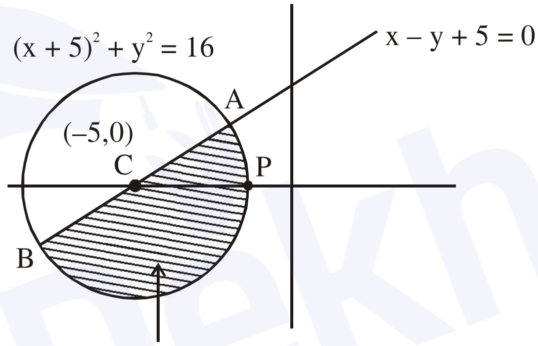
Sol. $|z + 5| \le 4$

$(x + 5)^2 + y^2 \le 16$ .... (1)

$z(1+i) + \bar{z}(1-i) \ge -10$

$(z + \bar{z}) + i(z - \bar{z}) \ge -10$

$x - y + 5 \ge 0$ .... (2)



Region bounded by inequalities (1) & (2)

$|z + 1|^2 = |z - (-1)|^2$

Let P(–1, 0)

$\boxed{|z+1|^2_{Max.} = PB^2}$ (where B is in 3rd quadrant)

for point of intersection

$$\left.\begin{matrix}(x+5)^2 + y^2 = 16 \\ x - y + 5 = 0\end{matrix}\right\rangle y = \pm 2\sqrt{2}$$

$A\left(2\sqrt{2} - 5, 2\sqrt{2}\right)$ $B\left(-2\sqrt{2} - 5, -2\sqrt{2}\right)$

$$PB^2 = \left(+2\sqrt{2} + 4\right)^2 + \left(2\sqrt{2}\right)^2$$

$$|z+1|^2 = 8 + 16 + 16\sqrt{2} + 8$$

$$\alpha + \beta\sqrt{2} = 32 + 16\sqrt{2}$$

$\alpha = 32$, $\beta = 16 \Rightarrow \alpha + \beta = 48$

2. Let the normals at all the points on a given curve pass through a fixed point (a, b). If the curve passes through (3, –3) and $\left(4, -2\sqrt{2}\right)$, and given that $a - 2\sqrt{2}\,b = 3$, then $(a^2 + b^2 + ab)$ is equal to ______.

Official Ans. by NTA (9)

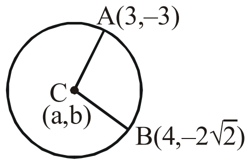
Sol. All normals of circle passes through centre

Radius = CA = CB

$CA^2 = CB^2$

$(a - 3)^2 + (b + 3)^2$

$= (a - 4)^2 + \left(b - 2\sqrt{2}\right)^2$



$a + \left(3 - 2\sqrt{2}\right)b = 3$

$a - 2\sqrt{2}\,b + 3b = 3$ ...(1)

given that $a - 2\sqrt{2}\,b = 3$ ...(2)

from (1) & (2) $\Rightarrow a = 3,\ b = 0$

$a^2 + b^2 + ab = 9$

3. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $p_n = (\alpha)^n + (\beta)^n$, $p_{n-1} = 11$ and $p_{n+1} = 29$ for some integer $n \geq 1$. Then, the value of p_n^2 is ______.

Official Ans. by NTA (324)

Sol. $x^2 - x - 1 = 0$ roots = α, β

$\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha^{n+1} = \alpha^n + \alpha^{n-1}$

$\beta^2 - \beta - 1 = 0 \Rightarrow \beta^{n+1} = \beta^n + \beta^{n-1}$

+

$P_{n+1} = P_n + P_{n-1}$

$29 = P_n + 11$

$P_n = 18$

$P_n^2 = 324$

4. If $I_{m,n} = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, for m, n ≥ 1 and $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}dx = \alpha I_{m,n}$, $\alpha \in R$, then α equals

Official Ans. by NTA (1)

Sol. $I_{m,n} = \int_0^1 x^{m-1}(1-x)^{n-1}dx = I_{n,m}$

Now Let $x = \frac{1}{y+1} \Rightarrow dx = -\frac{1}{(y+1)^2}dy$

so

$$I_{m,n} = -\int_\infty^0 \frac{1}{(y+1)^{m-1}}\frac{y^{n-1}}{(y+1)^{n-1}}\frac{dy}{(y+1)^2} = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

similarly $I_{m,n} = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$

Now $2I_{m,n} = \int_0^\infty \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}dy$

$$= \int_0^\infty \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}dy$$

$$= \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}dy + \underbrace{\int_1^\infty \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}dy}_{\text{substitute } y=\frac{1}{t}}$$

$$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}dy - \int_1^0 \frac{t^{n-1} + t^{m-1}}{t^{m+n-2}}\frac{t^{m+n}}{(1+t)^{m+n}}\frac{dt}{t^2}$$

$$\Rightarrow \text{Hence } 2I_{m,n} = 2\int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}dy \Rightarrow \alpha = 1$$

5. If the arithmetic mean and geometric mean of the p^{th} and q^{th} terms of the sequence –16, 8, –4, 2, ... satisfy the equation $4x^2 - 9x + 5 = 0$, then p + q is equal to ______.

Official Ans. by NTA (10)

Sol. $4x^2 - 9x + 5 = 0 \Rightarrow x = 1, \frac{5}{4}$

Now given $\frac{5}{4} = \frac{t_p + t_q}{2}, t = t_p t_q$ where

$$t_r = -16\left(-\frac{1}{2}\right)^{r-1}$$

so $\frac{5}{4} = -8\left[\left(-\frac{1}{2}\right)^{p-1} + \left(-\frac{1}{2}\right)^{q-1}\right]$

$$1 = 256\left(-\frac{1}{2}\right)^{p+q-2} \Rightarrow 2^{p+q-2} = (-1)^{p+q-2}\, 2^8$$

hence p + q = 10

6. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is ______.

Official Ans. by NTA (1000)

Sol. Let N be the four digit number

gcd(N,18) = 3

Hence N is an odd integer which is divisible by 3 but not by 9.

4 digit odd multiples of 3

1005, 1011,......., 9999 $\rightarrow$ 1500

4 digit odd multiples of 9

1017, 1035,......., 9999 $\rightarrow$ 500

Hence number of such N = 1000

7. Let L be a common tangent line to the curves $4x^2 + 9y^2 = 36$ and $(2x)^2 + (2y)^2 = 31$. Then the square of the slope of the line L is ______.

Official Ans. by NTA (3)

Sol. Given curves are $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$$x^2 + y^2 = \frac{31}{4}$$

let slope of common tangent be m

so tangents are $y = mx \pm \sqrt{9m^2 + 4}$

$$y = mx \pm \frac{\sqrt{31}}{2}\sqrt{1 + m^2}$$

hence $9m^2 + 4 = \frac{31}{4}(1 + m^2)$

$$\Rightarrow 36m^2 + 16 = 31 + 31m^2 \Rightarrow m^2 = 3$$

8. Let a be an integer such that all the real roots of the polynomial $2x^5+5x^4 + 10x^3 + 10x^2 + 10x + 10$ lie in the interval (a, a + 1). Then, |a| is equal to ______.

Official Ans. by NTA (2)

Sol. Let $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10 = f(x)$

Now $f(-2) = -34$ and $f(-1) = 3$

Hence $f(x)$ has a root in (–2,–1)

Further $f'(x) = 10x^4 + 20x^3 + 20x^2 + 20x + 10$

$$= 10x^2\left[\left(x^2 + \frac{1}{x^2}\right) + 2\left(x + \frac{1}{x}\right) + 20\right]$$

$$= 10x^2\left[\left(x + \frac{1}{x} + 1\right)^2 + 17\right] > 0$$

Hence $f(x)$ has only one real root, so |a| = 2

9. Let $X_1, X_2, \ldots, X_{18}$ be eighteen observations such that $\sum_{i=1}^{18}(X_i - \alpha) = 36$ and $\sum_{i=1}^{18}(X_i - \beta)^2 = 90$, where α and β are distinct real numbers. If the standard deviation of these observations is 1, then the value of $|\alpha - \beta|$ is ______.

Official Ans. by NTA (4)

Sol. $\sum_{i=1}^{18}(x_i-\alpha)=36, \sum_{i=1}^{18}(x_i-\beta)^2=90$

$\Rightarrow \sum_{i=1}^{18}x_i=18(\alpha+2), \sum_{i=1}^{18}x_i^2-2\beta\sum_{i=1}^{18}x_i+18\beta^2=90$

Hence $\sum x_i^2=90-18\beta^2+36\beta(\alpha+2)$

Given $\frac{\sum x_i^2}{18}-\left(\frac{\sum x_i}{18}\right)^2=1$

$\Rightarrow 90-18\beta^2+36\beta(\alpha+2)-18(\alpha+2)^2=18$

$\Rightarrow 5-\beta^2+2\alpha\beta+4\beta-\alpha^2-4\alpha-4=1$

$\Rightarrow (\alpha-\beta)^2+4(\alpha-\beta)=0 \Rightarrow |\alpha-\beta|=0$ or 4

As α and β are distinct $|\alpha-\beta|=4$

10. If the matrix $A=\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}$ satisfies the equation $A^{20}+\alpha A^{19}+\beta A=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}$ for some real numbers α and β, then $\beta-\alpha$ is equal to ______.

Official Ans. by NTA (4)

Sol. $A=\begin{bmatrix}1&0&0\\0&2&0\\3&0&-1\end{bmatrix}$

$A^2=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}, A^3=\begin{bmatrix}1&0&0\\0&8&0\\3&0&-1\end{bmatrix}$

$A^4=\begin{bmatrix}1&0&0\\0&16&0\\0&0&1\end{bmatrix}$

Hence

$A^{20}=\begin{bmatrix}1&0&0\\0&2^{20}&0\\0&0&1\end{bmatrix}, A^{19}=\begin{bmatrix}1&0&0\\0&2^{19}&0\\3&0&-1\end{bmatrix}$

So $A^{20}+\alpha A^{19}+\beta A=\begin{bmatrix}1+\alpha+\beta&0&0\\0&2^{20}+\alpha.2^{19}+2\beta&0\\3\alpha+3\beta&0&1-\alpha-\beta\end{bmatrix}$

$=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}$

Therefore $\alpha+\beta=0$ and $2^{20}+2^{19}\alpha-2\alpha=4$

$\Rightarrow \alpha=\frac{4(1-2^{18})}{2(2^{18}-1)}=-2$

hence $\beta=2$

so $(\beta-\alpha)=4$