MATHEMATICS

SECTION-A

1. The point P (a,b) undergoes the following three transformations successively :

(a) reflection about the line y = x.

(b) translation through 2 units along the positive direction of x-axis.

(c) rotation through angle $\frac{\pi}{4}$ about the origin in the anti-clockwise direction.

If the co-ordinates of the final position of the point P are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$, then the value of 2a + b is equal to :

(1) 13 (2) 9 (3) 5 (4) 7

Official Ans. by NTA (2)

Sol. Image of A(a,b) along y = x is B(b,a). Translating it 2 units it becomes C(b + 2, a).

Now, applying rotation theorem

$$-\frac{1}{2}+\frac{7}{\sqrt{2}}i=((b+2)+ai)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$\frac{-1}{\sqrt{2}}+\frac{7}{\sqrt{2}}i=\left(\frac{b+2}{\sqrt{2}}-\frac{a}{\sqrt{2}}\right)+i\left(\frac{b+2}{\sqrt{2}}+\frac{a}{\sqrt{2}}\right)$$

$\Rightarrow b - a + 2 = -1$ ....(i)

and $b + 2 + a = 7$ ....(ii)

$\Rightarrow a = 4; b = 1$

$\Rightarrow 2a + b = 9$

2. A possible value of 'x', for which the ninth term in the expansion of $\left\{3^{\log_3\sqrt{25^{x-1}+7}}+3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}\right\}^{10}$ in the increasing powers of $3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}$ is equal to 180, is :

(1) 0 (2) –1 (3) 2 (4) 1

Official Ans. by NTA (4)

Sol. ${}^{10}C_8\left(25^{(x-1)}+7\right)\times\left(5^{(x-1)}+1\right)^{-1}=180$

$$\Rightarrow \frac{25^{x-1}+7}{5^{(x-1)}+1}=4$$

$$\Rightarrow \frac{t^2+7}{t+1}=4;$$

$\Rightarrow t = 1, 3 = 5^{x-1}$

$\Rightarrow x - 1 = 0$ (one of the possible value).

$\Rightarrow x = 1$

3. For real numbers α and $\beta \neq 0$, if the point of intersection of the straight lines

$$\frac{x-\alpha}{1}=\frac{y-1}{2}=\frac{z-1}{3} \text{ and } \frac{x-4}{\beta}=\frac{y-6}{3}=\frac{z-7}{3},$$

lies on the plane $x + 2y - z = 8$, then $\alpha - \beta$ is equal to :

(1) 5 (2) 9 (3) 3 (4) 7

Official Ans. by NTA (4)

Sol. First line is $(\phi + \alpha, 2\phi + 1, 3\phi + 1)$

and second line is $(q\beta + 4, 3q + 6, 3q + 7)$.

For intersection $\phi + \alpha = q\beta + 4$ ...(i)

$2\phi + 1 = 3q + 6$ ...(i)

$3\phi + 1 = 3q + 7$ ...(iii)

for (ii) & (iii) $\phi = 1, q = -1$

So, from (i) $\alpha + \beta = 3$

Now, point of intersection is $(\alpha + 1, 3, 4)$

It lies on the plane.

Hence, $\alpha = 5$ & $\beta = -2$

4. Let f : **R** □ **R** be defined as

$f(x + y) + f(x - y) = 2 f(x) f(y)$, $f\left(\frac{1}{2}\right)=-1$. Then, the value of $\sum_{k=1}^{20}\frac{1}{\sin(k)\sin(k+f(k))}$ is equal to :

(1) $\text{cosec}^2(21)\cos(20)\cos(2)$

(2) $\sec^2(1)\sec(21)\cos(20)$

(3) $\text{cosec}^2(1)\text{cosec}(21)\sin(20)$

(4) $\sec^2(21)\sin(20)\sin(2)$

Official Ans. by NTA (3)

Sol. $f(x) = \cos \lambda x$

$$\because\ f\left(\frac{1}{2}\right) = -1$$

$$\text{So, } -1 = \cos\frac{\lambda}{2}$$

$$\Rightarrow\ \lambda = 2\pi$$

Thus $f(x) = \cos 2\pi x$

Now k is natural number

Thus $f(k) = 1$

$$\sum_{k=1}^{20} \frac{1}{\sin k \sin(k+1)} = \frac{1}{\sin 1}\sum_{k=1}^{20}\left[\frac{\sin((k+1)-k)}{\sin k \cdot \sin(k+1)}\right]$$

$$= \frac{1}{\sin 1}\sum_{k=1}^{20}(\cot k - \cot(k+1))$$

$$= \frac{\cot 1 - \cot 21}{\sin 1} = \operatorname{cosec}^2 1 \operatorname{cosec}(21) \cdot \sin 20$$

5. Let $\mathbb{C}$ be the set of all complex numbers. Let

$S_1 = \{z \in \mathbb{C} : |z-2| \le 1\}$ and

$S_2 = \{z \in \mathbb{C} : z(1+i) + \bar{z}(1-i) \ge 4\}$.

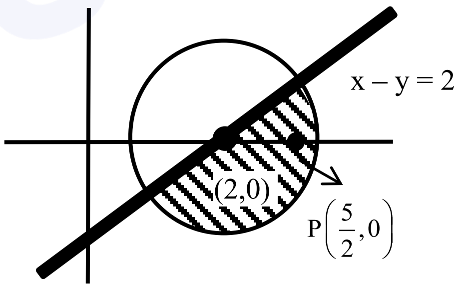
Then, the maximum value of $\left|z - \frac{5}{2}\right|^2$ for $z \in S_1 \cap S_2$ is equal to :

(1) $\frac{3+2\sqrt{2}}{4}$ (2) $\frac{5+2\sqrt{2}}{2}$

(3) $\frac{3+2\sqrt{2}}{2}$ (4) $\frac{5+2\sqrt{2}}{4}$

Official Ans. by NTA (4)

Sol. $|t-2| \le 1$ Put $t = x + iy$



$(x-2)^2 + y^2 \le 1$

Also, $t(1+i) + \bar{t}(1-i) \ge 4$

Gives $x - y \ge 2$

Let point on circle be $A(2 + \cos\theta, \sin\theta)$

$$\theta \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$$

$$(AP)^2 = \left(2 + \cos\theta - \frac{5}{2}\right)^2 + \sin^2\theta$$

$$= \cos^2\theta - \cos\theta + \frac{1}{4} + \sin^2\theta$$

$$= \frac{5}{4} - \cos\theta$$

For $(AP)^2$ maximum $\theta = -\frac{3\pi}{4}$

$$(AP)^2 = \frac{5}{4} + \frac{1}{\sqrt{2}} = \frac{5\sqrt{2}+4}{4\sqrt{2}}$$

6. A student appeared in an examination consisting of 8 true–false type questions. The student guesses the answers with equal probability. The smallest value of n, so that the probability of guessing at least 'n' correct answers is less than $\frac{1}{2}$, is :

(1) 5   (2) 6   (3) 3   (4) 4

Official Ans. by NTA (1)

Sol. $P(E) < \frac{1}{2}$

$$\Rightarrow \sum_{r=n}^{8} {}^8C_r \left(\frac{1}{2}\right)^{8-r}\left(\frac{1}{2}\right)^r < \frac{1}{2}$$

$$\Rightarrow \sum_{r=n}^{8} {}^8C_r \left(\frac{1}{2}\right)^8 < \frac{1}{2}$$

$$\Rightarrow {}^8C_n + {}^8C_{n+1} + \dots + {}^8C_8 < 128$$

$$\Rightarrow 256 - \left({}^8C_0 + {}^8C_1 + \dots + {}^8C_{n-1}\right) < 128$$

$$\Rightarrow {}^8C_0 + {}^8C_1 + \dots + {}^8C_{n-1} > 128$$

$$\Rightarrow\ n - 1 \ge 4$$

$$\Rightarrow\ n \ge 5$$

7. If $\tan\left(\frac{\pi}{9}\right), x, \tan\left(\frac{7\pi}{18}\right)$ are in arithmetic progression and $\tan\left(\frac{\pi}{9}\right), y, \tan\left(\frac{5\pi}{18}\right)$ are also in arithmetic progression, then $|x - 2y|$ is equal to :

(1) 4   (2) 3   (3) 0   (4) 1

Official Ans. by NTA (3)

Sol. $x=\frac{1}{2}\left(\tan\frac{\pi}{9}+\tan\frac{7\pi}{18}\right)$

and $2y=\tan\frac{\pi}{9}+\tan\frac{5\pi}{18}$

so, $x-2y=\frac{1}{2}\left(\tan\frac{\pi}{9}+\tan\frac{7\pi}{18}\right)$

$-\left(\tan\frac{\pi}{9}+\tan\frac{5\pi}{18}\right)$

$\Rightarrow |x-2y| = \left|\frac{\cot\frac{\pi}{9}-\tan\frac{\pi}{9}}{2}-\tan\frac{5\pi}{18}\right|$

$=\left|\cot\frac{2\pi}{9}-\cot\frac{2\pi}{9}\right|=0$

$\left(\text{as }\tan\frac{5\pi}{18}=\cot\frac{2\pi}{9};\tan\frac{7\pi}{18}=\cot\frac{\pi}{9}\right)$

8. Let the mean and variance of the frequency distribution

x :	$x_1 = 2$	$x_2 = 6$	$x_3 = 8$	$x_4 = 9$
f :	4	4	α	β

be 6 and 6.8 respectively. If x_3 is changed from 8 to 7, then the mean for the new data will be:

(1) 4 (2) 5

(3) $\frac{17}{3}$ (4) $\frac{16}{3}$

Official Ans. by NTA (3)

Sol. Given $32 + 8\alpha + 9\beta = (8 + \alpha + \beta) \times 6$

$\Rightarrow 2\alpha + 3\beta = 16$ ...(i)

Also, $4 \times 16 + 4 \times \alpha + 9\beta = (8 + \alpha + \beta) \times 6.8$

$\Rightarrow 640 + 40\alpha + 90\beta = 544 + 68\alpha + 68\beta$

$\Rightarrow 28\alpha - 22\beta = 96$

$\Rightarrow 14\alpha - 11\beta = 48$ ...(ii)

from (i) & (ii)

$\alpha = 5$ & $\beta = 2$

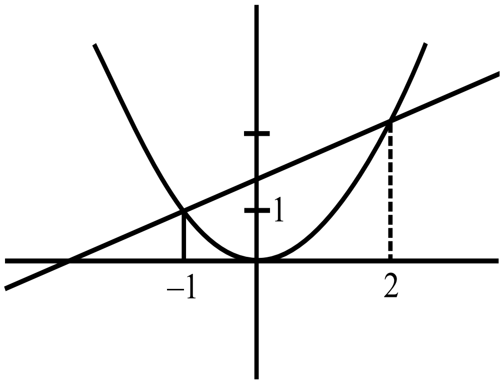
so, new mean $=\frac{32+35+18}{15}=\frac{85}{15}=\frac{17}{3}$

9. The area of the region bounded by $y - x = 2$ and $x^2 = y$ is equal to :-

(1) $\frac{16}{3}$ (2) $\frac{2}{3}$ (3) $\frac{9}{2}$ (4) $\frac{4}{3}$

Official Ans. by NTA (3)

Sol.



$y - x = 2$, $x^2 = y$

Now, $x^2 = 2 + x$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x + 1)(x - 2) = 0$

Area $=\int_{-1}^{2}(2+x-x^2)$

$=\left|2x+\frac{x^2}{2}-\frac{x^3}{3}\right|_{-1}^{2}$

$=\left(4+2-\frac{8}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right)$

$=6-3+2-\frac{1}{2}=\frac{9}{2}$

10. Let $y = y(x)$ be the solution of the differential equation $(x - x^3)dy = (y + yx^2 - 3x^4)dx$, $x > 2$. If $y(3) = 3$, then $y(4)$ is equal to :

(1) 4 (2) 12 (3) 8 (4) 16

Official Ans. by NTA (2)

Sol. $(x - x^3)dy = (y + yx^2 - 3x^4)dx$

$\Rightarrow xdy - ydx = (yx^2 - 3x^4)dx + x^3dy$

$\Rightarrow \frac{xdy-ydx}{x^2}=(ydx+xdy)-3x^2dx$

$\Rightarrow d\left(\frac{y}{x}\right)=d(xy)-d(x^3)$

Integrate

$\Rightarrow \frac{y}{x}=xy-x^3+c$

given $f(3) = 3$

$\Rightarrow \frac{3}{3} = 3 \times 3 - 3^3 + c$

$\Rightarrow c = 19$

$\therefore \frac{y}{x} = xy - x^3 + 19$

at $x = 4$, $\frac{y}{4} = 4y - 64 + 19$

$15y = 4 \times 45$

$\Rightarrow y = 12$

11. The value of $\lim_{x \to 0}\left(\frac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}}\right)$ is equal to :

(1) 0 (2) 4 (3) –4 (4) –1

Official Ans. by NTA (3)

Sol. $\lim_{x\to 0}\left(\frac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}}\right)$

$= \lim_{x\to 0}\left(\frac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}}\right)$

$= \lim_{x\to 0}\left(\frac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}}\right)$

$\left(\frac{\left(\sqrt[8]{1-\sin x} + \sqrt[8]{1+\sin x}\right)}{\sqrt[8]{1-\sin x} + \sqrt[8]{1+\sin x}}\right)$

$\left(\frac{\left(\sqrt[4]{1-\sin x} + \sqrt[4]{1+\sin x}\right)}{\sqrt[4]{1-\sin x} + \sqrt[4]{1+\sin x}}\right)$

$\left(\frac{\left(\sqrt[2]{1-\sin x} + \sqrt[2]{1+\sin x}\right)}{\sqrt[2]{1-\sin x} + \sqrt[2]{1+\sin x}}\right)$

$= \lim_{x\to 0}\left(\frac{x}{1-\sin x - (1+\sin x)}\right)$

$\left(\sqrt[8]{1-\sin x} + \sqrt[8]{1+\sin x}\right)\left(\sqrt[4]{1-\sin x} + \sqrt[4]{1+\sin x}\right)$

$\left(\sqrt[2]{1-\sin x} + \sqrt[2]{1+\sin x}\right)$

$= \lim_{x\to 0}\frac{x}{(-2\sin x)}\left(\sqrt[8]{1-\sin x} + \sqrt[8]{1+\sin x}\right)$

$\left(\sqrt[4]{1-\sin x} + \sqrt[4]{1+\sin x}\right)\left(\sqrt[2]{1-\sin x} + \sqrt[2]{1+\sin x}\right)$

$= \lim_{x\to 0}\left(-\frac{1}{2}\right)(2)\,(2)\,(2)\ \left\{\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right\} = -4$

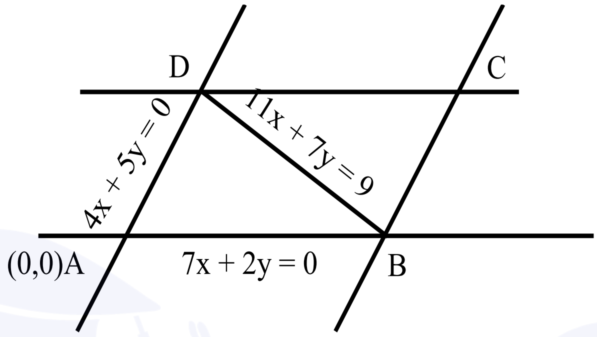
12. Two sides of a parallelogram are along the lines $4x + 5y = 0$ and $7x + 2y = 0$. If the equation of one of the diagonals of the parallelogram is $11x + 7y = 9$, then other diagonal passes through the point :

(1) (1,2) (2) (2,2)

(3) (2,1) (4) (1,3)

Official Ans. by NTA (2)

Sol. Both the lines pass through origin.



point D is equal of intersection of $4x + 5y = 0$ & $11x + 7y = 9$

So, coordinates of point $D = \left(\frac{5}{3}, -\frac{4}{3}\right)$

Also, point B is point of intersection of $7x + 2y = 0$ & $11x + 7y = 9$

So, coordinates of point $B = \left(-\frac{2}{3}, \frac{7}{3}\right)$

diagonals of parallelogram intersect at middle

let middle point of B,D

$\Rightarrow \left(\frac{\frac{5}{3}-\frac{2}{3}}{2}, \frac{\frac{-4}{3}+\frac{7}{3}}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$

equation of diagonal AC

$\Rightarrow (y-0) = \frac{\frac{1}{\alpha}-0}{\frac{1}{\alpha}-0}(\pi - 0)$

$y = x$

diagonal AC passes through (2, 2).

13. Let $\alpha = \max\limits_{x\in\mathbf{R}}\{8^{2\sin 3x}\cdot 4^{4\cos 3x}\}$ and $\beta = \min\limits_{x\in\mathbf{R}}\{8^{2\sin 3x}\cdot 4^{4\cos 3x}\}$. If $8x^2 + bx + c = 0$ is a quadratic equation whose roots are $\alpha^{1/5}$ and $\beta^{1/5}$, then the value of $c - b$ is equal to :

(1) 42 (2) 47 (3) 43 (4) 50

Official Ans. by NTA (1)

Sol. $\alpha = \max\{8^{2\sin 3x}\cdot 4^{4\cos 3x}\}$

$= \max\{2^{6\sin 3x}\cdot 2^{8\cos 3x}\}$

$= \max\{2^{6\sin 3x+8\cos 3x}\}$

and $\beta = \min\{8^{2\sin 3x}\cdot 4^{4\cos 3x}\} = \min\{2^{6\sin 3x+8\cos 3x}\}$

Now range of 6 sin 3x + 8 cos 3x

$= \left[-\sqrt{6^2+8^2}, +\sqrt{6^2+8^2}\right] = [-10,10]$

$\alpha = 2^{10}$ & $\beta = 2^{-10}$

So, $\alpha^{1/5} = 2^2 = 4$

$\Rightarrow \beta^{1/5} = 2^{-2} = 1/4$

quadratic $8x^2 + bx + c = 0$, $c - b =$

$8\times[(\text{product of roots}] + (\text{sum of roots})$

$= 8\times\left[4\times\frac{1}{4}+4+\frac{1}{4}\right] = 8\times\left[\frac{21}{4}\right] = 42$

14. Let $f : [0, \infty) \to [0, 3]$ be a function defined by

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x\}, & 0 \le x \le \pi \\ 2 + \cos x, & x > \pi \end{cases}$$

Then which of the following is true ?

(1) f is continuous everywhere but not differentiable exactly at one point in $(0, \infty)$

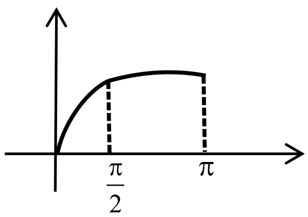
(2) f is differentiable everywhere in $(0, \infty)$

(3) f is not continuous exactly at two points in $(0, \infty)$

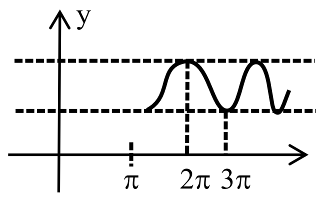
(4) f is continuous everywhere but not differentiable exactly at two points in $(0, \infty)$

Official Ans. by NTA (2)

Sol. Graph of $\max\{\sin t : 0 \le t \le x\}$ in $x \in [0, \pi]$

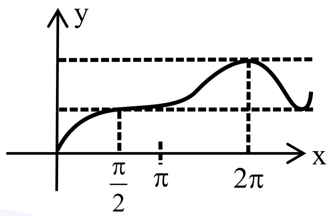


& graph of cos for $x \in [\pi, \infty)$



So graph of

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x, & 0 \le x \le \pi \\ 2 + \cos x & x > h \end{cases}$$



f(x) is differentiable everywhere in $(0,\infty)$

15. Let **N** be the set of natural numbers and a relation R on **N** be defined by

$R = \{(x,y) \in \mathbf{N}\times\mathbf{N} : x^3 - 3x^2y - xy^2 + 3y^3 = 0\}$.

Then the relation R is :

(1) symmetric but neither reflexive nor transitive

(2) reflexive but neither symmetric nor transitive

(3) reflexive and symmetric, but not transitive

(4) an equivalence relation

Official Ans. by NTA (2)

Sol. $x^3 - 3x^2y - xy^2 + 3y^3 = 0$

$\Rightarrow x(x^2 - y^2) - 3y(x^2 - y^2) = 0$

$\Rightarrow (x - 3y)(x - y)(x + y) = 0$

Now, $x = y \ \forall (x,y) \in N\times N$ so reflexive

But not symmetric & transitive

See, (3,1) satisfies but (1,3) does not. Also (3,1) & (1,–1) satisfies but (3, –1) does not

16. Which of the following is the negation of the statement "for all M > 0, there exists $x \in S$ such that $x \ge M$" ?

(1) there exists $M > 0$, such that $x < M$ for all $x \in S$

(2) there exists $M > 0$, there exists $x \in S$ such that $x \ge M$

(3) there exists $M > 0$, there exists $x \in S$ such that $x < M$

(4) there exists $M > 0$, such that $x \ge M$ for all $x \in S$

Official Ans. by NTA (1)

Sol. P : for all $M > 0$, there exists $x \in S$ such that $x \geq M$.

$\sim P$: there exists $M > 0$, for all $x \in S$

Such that $x < m$

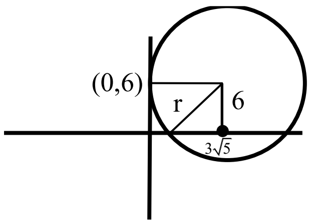
Negation of 'there exsits' is 'for all'.

17. Consider a circle C which touches the y-axis at (0, 6) and cuts off an intercept $6\sqrt{5}$ on the x-axis. Then the radius of the circle C is equal to :

(1) $\sqrt{53}$ (2) 9 (3) 8 (4) $\sqrt{82}$

Official Ans. by NTA (2)

Sol.



$r = \sqrt{6^2 + (3\sqrt{5})^2}$

$= \sqrt{36 + 45} = 9$

18. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} = \vec{b} \times (\vec{b} \times \vec{c})$. If magnitudes of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are $\sqrt{2}, 1$ and 2 respectively and the angle between $\vec{b}$ and $\vec{c}$ is $\theta \left(0 < \theta < \frac{\pi}{2}\right)$, then the value of $1 + \tan\theta$ is equal to :

(1) $\sqrt{3} + 1$ (2) 2

(3) 1 (4) $\frac{\sqrt{3}+1}{\sqrt{3}}$

Official Ans. by NTA (2)

Sol. $\vec{a} = (\vec{b} \cdot \vec{c})\vec{b} - (\vec{b}.\vec{b})\vec{c}$

$= 1.2\cos\theta\vec{b} - \vec{c}$

$\Rightarrow \vec{a} = 2\cos\theta\vec{b} - \vec{c}$

$|\vec{a}|^2 = (2\cos\theta)^2 + 2^2 - 2.2\cos\theta\vec{b} \cdot \vec{c}$

$\Rightarrow 2 = 4\cos^2\theta + 4 - 4\cos\theta \cdot 2\cos\theta$

$\Rightarrow -2 = -4\cos^2\theta$

$\Rightarrow \cos^2\theta = \frac{1}{2}$

$\Rightarrow \sec^2\theta = 2$

$\Rightarrow \tan^2\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4}$

$1 + \tan\theta = 2$.

19. Let A and B be two 3 × 3 real matrices such that $(A^2 - B^2)$ is invertible matrix. If $A^5 = B^5$ and $A^3B^2 = A^2B^3$, then the value of the determinant of the matrix A^3+B^3 is equal to :

(1) 2 (2) 4 (3) 1 (4) 0

Official Ans. by NTA (4)

Sol. $C = A^2 - B^2; |C| \neq 0$

$A^5 = B^5$ and $A^3B^2 = A^2B^2$

Now, $A^5 - A^3B^2 = B^5 - A^2B^3$

$\Rightarrow A^3(A^2 - B^2) + B^3(A^2 - B^2) = 0$

$\Rightarrow (A^3 + B^3)(A^2 - B^2) = 0$

Post multiplying inverse of $A^2 - B^2$:

$A^3 + B^3 = 0$

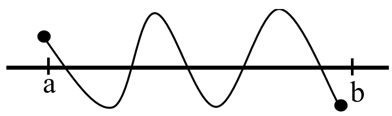
20. Let f : (a,b) → **R** be twice differentiable function such that $f(x) = \int_a^x g(t)\,dt$ for a differentiable function g(x). If $f(x) = 0$ has exactly five distinct roots in (a, b), then $g(x)g'(x) = 0$ has at least :

(1) twelve roots in (a, b) (2) five roots in (a, b)

(3) seven roots in (a, b) (4) three roots in (a, b)

Official Ans. by NTA (3)

Sol.



$f(x) = \int_a^x g(t)dt$

$f(x) \rightarrow 5$

$g(x) \rightarrow 4$

$g'(x) \rightarrow 3$

SECTION-B

1. Let $\vec{a} = \hat{i} - \alpha\hat{j} + \beta\hat{k}$, $\vec{b} = 3\hat{i} + \beta\hat{j} - \alpha\hat{k}$ and $\vec{c} = -\alpha\hat{i} - 2\hat{j} + \hat{k}$, where α and β are integers. If $\vec{a} \cdot \vec{b} = -1$ and $\vec{b} \cdot \vec{c} = 10$, then $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is equal to____ .

Official Ans. by NTA (9)

Sol. $\vec{a} = (1, -\alpha, \beta)$

$\vec{b} = (3, \beta, -\alpha)$

$\vec{c} = (-\alpha, -2, 1); \alpha, \beta \in I$

$\vec{a}.\vec{b} = -1 \Rightarrow 3 - \alpha\beta - \alpha\beta = -1$

$\Rightarrow \alpha\beta = 2$

1	2
2	1
−1	−2
−2	−1

$\vec{b} \cdot \vec{c} = 10$

$\Rightarrow -3\alpha - 2\beta - \alpha = 10$

$\Rightarrow 2\alpha + \beta + 5 = 0$

$\therefore \alpha = -2; \beta = -1$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 1 \end{vmatrix}$$

$= 1(-1 + 4) - 2(3 - 4) - 1(-6 + 2)$

$= 3 + 2 + 4 = 9$

2. The distance of the point P(3, 4, 4) from the point of intersection of the line joining the points. Q(3, –4, –5) and R(2, –3, 1) and the plane $2x + y + z = 7$, is equal to______.

Official Ans. by NTA (7)

Sol. $\overleftrightarrow{QR} : -\frac{x-3}{1} = \frac{y+4}{-1} = \frac{z+5}{-6} = r$

$\Rightarrow (x,y,z) \equiv (r + 3, -r - 4, -6r - 5)$

Now, satisfying it in the given plane.

We get $r = -2$.

so, required point of intersection is T(1,–2,7).

Hence, PT = 7.

3. If the real part of the complex number $z = \frac{3 + 2i\cos\theta}{1 - 3i\cos\theta}$, $\theta \in \left(0, \frac{\pi}{2}\right)$ is zero, then the value of $\sin^2 3\theta + \cos^2\theta$ is equal to______.

Official Ans. by NTA (1)

Sol. $\text{Re}(z) = \frac{3 - 6\cos^2\theta}{1 + 9\cos^2\theta} = 0$

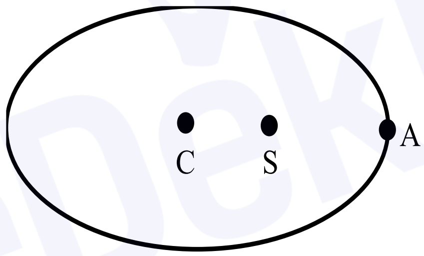
$\Rightarrow \theta = \frac{\pi}{4}$

Hence, $\sin^2 3\theta + \cos^2\theta = 1$.

4. Let E be an ellipse whose axes are parallel to the co-ordinates axes, having its center at (3, – 4), one focus at (4, – 4) and one vertex at (5, – 4). If $mx - y = 4$, $m > 0$ is a tangent to the ellipse E, then the value of $5m^2$ is equal to______.

Official Ans. by NTA (3)

Sol. Given C(3,–4), S(4,–4)



and A(5,–4)

Hence, a = 2 & ae = 1

$\Rightarrow e = \frac{1}{2}$

$\Rightarrow b^2 = 3$.

So, $E : \frac{(x-3)^2}{4} + \frac{(y+4)^2}{3} = 1$

Intersecting with given tangent.

$\frac{x^2 - 6x + 9}{4} + \frac{m^2x^2}{3} = 1$

Now, D = 0 (as it is tangent)

So, $5m^2 = 3$.

5. If $\int_0^{\pi} (\sin^3 x) e^{-\sin^2 x} dx = \alpha - \frac{\beta}{e}\int_0^1 \sqrt{t}\, e^t dt$, then $\alpha + \beta$ is equal to _____ .

Official Ans. by NTA (5)

Sol. $I = 2\int_0^{\pi/2} \sin^3 x\, e^{-\sin^2 x} dx$

$= 2\int_0^{\pi/2} \sin x\, e^{-\sin^2 x} dx + \int_0^{\pi/2} \underset{I}{\cos x}\; \underbrace{e^{-\sin^2 x}(-\sin 2x)}_{II} dx$

$= 2\int_0^{\pi/2} \sin x e^{-\sin^2 x} dx + \left[\cos x\, e^{-\sin^2 x}\right]_0^{\pi/2}$

$+ \int_0^{\pi/2} \sin x\, e^{-\sin^2 x}\, dx$

$= 3\int_0^{\pi/2} \sin x\, e^{-\sin^2 x} dx - 1$

$= \frac{3}{2}\int_{-1}^{0} \frac{e^{\alpha} d\alpha}{\sqrt{1+\alpha}} - 1$ (Put $-\sin^2 x = t$)

$= \frac{3}{2e}\int_0^1 \frac{e^x}{\sqrt{x}} dx - 1$ (put $1 + \alpha = x$)

$= \frac{3}{2e}\int_0^1 \underset{II}{e^x}\, \underset{II}{\frac{1}{\sqrt{x}}} dx - 1$

$= 2 - \frac{3}{e}\int_0^1 e^x \sqrt{x}\, dx$

Hence, $\alpha + \beta = \boxed{5}$

6. The number of real roots of the equation $e^{4x} - e^{3x} - 4e^{2x} - e^x + 1 = 0$ is equal to____ .

Official Ans. by NTA (2)

Sol. $t^4 - t^3 - 4t^2 - t + 1 = 0,\ e^x = t > 0$

$\Rightarrow t^2 - t - 4 - \frac{1}{t} + \frac{1}{t^2} = 0$

$\Rightarrow \alpha^2 - \alpha - 6 = 0,\ \alpha = t + \frac{1}{t} \geq 2$

$\Rightarrow \alpha = 3, -2$ (reject)

$\Rightarrow t + \frac{1}{t} = 3$

$\Rightarrow$ The number of real roots = 2

7. Let y= y(x) be the solution of the differential equation $dy = e^{\alpha x + y} dx$; $\alpha \in \mathbf{N}$. If $y(\log_e 2) = \log_e 2$ and $y(0) = \log_e\left(\frac{1}{2}\right)$, then the value of α is equal to____.

Official Ans. by NTA (2)

Sol. $\int e^{-y} dy = \int e^{\alpha x}\, dx$

$\Rightarrow e^{-y} = \frac{e^{\alpha x}}{\alpha} + c$ ...(i)

Put (x,y) = (ℓn2, ℓn2)

$\frac{-1}{2} = \frac{2^{\alpha}}{\alpha} + C$ ...(ii)

Put (x,y) $\equiv (0, -\ell n2)$ in (i)

$-2 = \frac{1}{\alpha} + C$ ...(iii)

(ii) – (iii)

$\frac{2^{\alpha} - 1}{\alpha} = \frac{3}{2}$

$\Rightarrow \alpha = 2$ (as $\alpha \in \mathbb{N}$)

8. Let n be a non-negative integer. Then the number of divisors of the form "4n + 1" of the number $(10)^{10}.(11)^{11}.(13)^{13}$ is equal to_____ .

Official Ans. by NTA (924)

Sol. $N = 2^{10} \times 5^{10} \times 11^{11} \times 13^{13}$

Now, power of 2 must be zero,

power of 5 can be anything,

power of 13 can be anything.

But, power of 11 should be even.

So, required number of divisors is

$1 \times 11 \times 14 \times 6 = 924$

9. Let $A = \{n \in \mathbf{N} \mid n^2 \leq n + 10{,}000\}$, $B = \{3k + 1 \mid k \in \mathbf{N}\}$ and $C = \{2k \mid k \in \mathbf{N}\}$, then the sum of all the elements of the set $A \cap (B - C)$ is equal to_______ .

Official Ans. by NTA (832)

Sol. $B - C \equiv \{7, 13, 19, \ldots 97, \ldots.\}$

Now, $n^2 - n \leq 100 \times 100$

$\Rightarrow n(n - 1) \leq 100 \times 100$

$\Rightarrow A = \{1, 2, \ldots., 100\}$.

So, $A \cap (B - C) = \{7, 13, 19, \ldots, 97\}$

Hence, sum $= \frac{16}{2}(7 + 97) = 832$

10. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $M = A + A^2 + A^3 + \ldots + A^{20}$, then the sum of all the elements of the matrix M is equal to______.

Official Ans. by NTA (2020)

Sol. $A^n = \begin{bmatrix} 1 & n & \frac{n^2+n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$

So, required sum

$$= 20 \times 3 + 2 \times \left(\frac{20 \times 21}{2}\right) + \sum_{r=1}^{20}\left(\frac{r^2+r}{2}\right)$$

$$= 60 + 420 + 105 + 35 \times 41 = 2020$$