MATHEMATICS

SECTION-A

1. If the mean and variance of the following data: 6, 10, 7, 13, a, 12, b, 12

are 9 and $\frac{37}{4}$ respectively, then $(a-b)^2$ is equal to:

(1) 24 (2) 12 (3) 32 (4) 16

Official Ans. by NTA (4)

Sol. Mean $=\frac{6+10+7+13+a+12+b+12}{8}=9$

$60 + a + b = 72$

$a + b = 12$ ...(1)

variance $=\frac{\sum x_i^2}{n}-\left(\frac{\sum x_i}{n}\right)^2=\frac{37}{4}$

$\Sigma x_i^2=6^2+10^2+7^2+13^2+a^2+b^2+12^2+12^2$

$= a^2 + b^2 + 642$

$\frac{a^2+b^2+642}{8}-(9)^2=\frac{37}{4}$

$\frac{a^2+b^2}{8}+\frac{321}{4}-81=\frac{37}{4}$

$\frac{a^2+b^2}{8}=81+\frac{37}{4}-\frac{321}{4}$

$\frac{a^2+b^2}{8}=81-71$

$\therefore\ a^2 + b^2 = 80$ ...(2)

From (1) $a^2 + b^2 + 2ab = 144$

$80 + 2ab = 144$ $\therefore\ 2ab = 64$

$(a-b)^2 = a^2 + b^2 - 2ab = 80 - 64 = 16$

2. The value of $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{(2j-1)+8n}{(2j-1)+4n}$ is equal to :

(1) $5+\log_e\left(\frac{3}{2}\right)$ (2) $2-\log_e\left(\frac{2}{3}\right)$

(3) $3+2\log_e\left(\frac{2}{3}\right)$ (4) $1+2\log_e\left(\frac{3}{2}\right)$

Official Ans. by NTA (4)

Sol. $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{\left(\frac{2j}{n}-\frac{1}{n}+8\right)}{\left(\frac{2j}{n}-\frac{1}{n}+4\right)}$

$\int_0^1\frac{2x+8}{2x+4}dx=\int_0^1 dx+\int_0^1\frac{4}{2x+4}dx$

$=1+4\frac{1}{2}\left(\ell n|2x+4|\right)\Big|_0^1$

$=1+2\ell n\left(\frac{3}{2}\right)$

3. Let $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+2\hat{j}+3\hat{k}$. Then the vector product $\left(\vec{a}+\vec{b}\right)\times\left(\left(\vec{a}\times\left(\left(\vec{a}-\vec{b}\right)\times\vec{b}\right)\right)\times\vec{b}\right)$ is equal to :

(1) $5\left(34\hat{i}-5\hat{j}+3\hat{k}\right)$ (2) $7\left(34\hat{i}-5\hat{j}+3\hat{k}\right)$

(3) $7\left(30\hat{i}-5\hat{j}+7\hat{k}\right)$ (4) $5\left(30\hat{i}-5\hat{j}+7\hat{k}\right)$

Official Ans. by NTA (2)

Sol. $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$

$\vec{b}=-\hat{i}+2\hat{j}+3\hat{k}$

$\vec{a}+\vec{b}=3\hat{j}+5\hat{k}$; $\vec{a}.\vec{b}=-1+2+6=7$

$\left(\left(\vec{a}\times\left(\left(\vec{a}-\vec{b}\right)\times\vec{b}\right)\right)\times\vec{b}\right)$

$\left(\left(\vec{a}\times\left(\vec{a}\times\vec{b}-\vec{b}\times\vec{b}\right)\right)\times\vec{b}\right)$

$\left(\vec{a}\times\left(\vec{a}\times\vec{b}-0\right)\right)\times\vec{b}$

$\left(\vec{a}\times\left(\vec{a}\times\vec{b}\right)\right)\times\vec{b}$

$\left(\left(\vec{a}.\vec{b}\right)\vec{a}-\left(\vec{a}.\vec{a}\right)\vec{b}\right)\times\vec{b}$

$\left(\vec{a}.\vec{b}\right)\vec{a}\times\vec{b}-\left(\vec{a}.\vec{a}\right)\left(\vec{b}\times\vec{b}\right)$

$\left(\vec{a}.\vec{b}\right)\left(\vec{a}\times\vec{b}\right)$

$\vec{a}\times\vec{b}=\begin{vmatrix}i & j & k\\ 1 & 1 & 2\\ -1 & 2 & 3\end{vmatrix}=-\hat{i}-5\hat{j}+3\hat{k}$

$\therefore\ 7\left(-\hat{i}-5\hat{j}+3\hat{k}\right)$

$\left(\vec{a}+\vec{b}\right)\times\left(7\left(-\hat{i}-5\hat{j}+3\hat{k}\right)\right)$

$7\left(0\hat{i}+3\hat{j}+5\hat{k}\right)\times\left(-\hat{i}-5\hat{j}+3\hat{k}\right)$

$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 3 & 5\\ -1 & -5 & 3\end{vmatrix}$

$\Rightarrow 34\hat{i}-(5)\hat{j}+\left(3\hat{k}\right)$

$\Rightarrow 34\hat{i}-5\hat{j}+3\hat{k}$

$\therefore\ 7\left(34\hat{i}-5\hat{j}+3\hat{k}\right)$

4. The value of the definite integral

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{x\cos x})(\sin^4 x+\cos^4 x)}$$

is equal to :

(1) $-\frac{\pi}{2}$ (2) $\frac{\pi}{2\sqrt{2}}$ (3) $-\frac{\pi}{4}$ (4) $\frac{\pi}{\sqrt{2}}$

Official Ans. by NTA (2)

Sol. $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{(1+e^{x\cos x})(\sin^4 x+\cos^4 x)}$ ...(1)

Using $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$$I=\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{-x\cos x})(\sin^4 x+\cos^4 x)}$$

Add (1) and (2)

$$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$$

$$2I=2\int_{0}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x+\cos^4 x}$$

$$I=\int_{0}^{\frac{\pi}{4}} \frac{(1+\tan^2 x)\sec^2 x}{\tan^4 x+1}dx$$

$$I=\int_{0}^{\frac{\pi}{4}} \frac{\left(1+\frac{1}{\tan^2 x}\right)\sec^2 x}{\left(\tan x-\frac{1}{\tan x}\right)^2+2}dx$$

$$\tan x-\frac{1}{\tan x}=t$$

$$\left(1+\frac{1}{\tan^2 x}\right)\sec^2 x dx=dt$$

$$I=\int_{-\infty}^{0} \frac{dt}{t^2+2}=\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_{-\infty}^{0}$$

$$I=0-\frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right)=\frac{\pi}{2\sqrt{2}}$$

5. Let C be the set of all complex numbers. Let

$S_1=\{z\in C \mid |z-3-2i|^2=8\}$,

$S_2=\{z\in C \mid \text{Re}(z)\geq 5\}$ and

$S_3=\{z\in C \mid |z-\bar{z}|\geq 8\}$.

Then the number of elements in $S_1\cap S_2\cap S_3$ is equal to

(1) 1 (2) 0 (3) 2 (4) Infinite

Official Ans. by NTA (1)

Sol. $S_1:|z-3-2i|^2=8$

$|z-3-2i|=2\sqrt{2}$

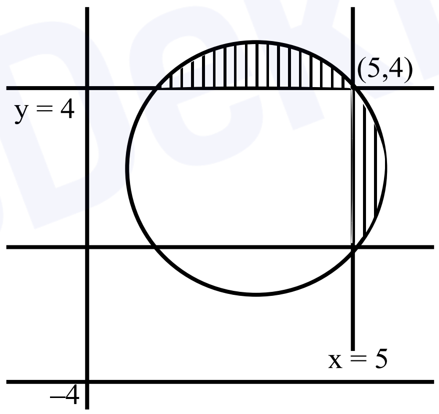
$(x-3)^2+(y-2)^2=(2\sqrt{2})^2$

$S_2: x\geq 5$

$S_3:|z-\bar{z}|\geq 8$

$|2iy|\geq 8$

$2|y|\geq 8 \therefore y\geq 4, y\leq -4$



$n(S_1\cap S_2\cap S_3)=1$

6. If the area of the bounded region

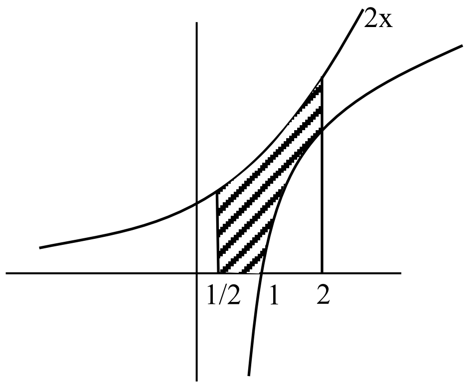
$$R=\left\{(x,y):\max\{0,\log_e x\}\leq y\leq 2^x, \frac{1}{2}\leq x\leq 2\right\}$$

is, $\alpha(\log_e 2)^{-1}+\beta(\log_e 2)+\gamma$, then the value of $(\alpha+\beta-2\gamma)^2$ is equal to :

(1) 8 (2) 2 (3) 4 (4) 1

Official Ans. by NTA (2)

Sol. $R=\left\{(x,y):\max(0,\log_e x)\le y\le 2^x,\frac{1}{2}\le x\le 2\right\}$



$$\int_{\frac{1}{2}}^{2}2^x\,dx-\int_{1}^{2}\ell nx\,dx$$

$$\Rightarrow\left[\frac{2^x}{\ln 2}\right]_{1/2}^{2}-[x\ln x-x]_1^2$$

$$\Rightarrow\frac{(2^2)-2^{1/2}}{\log_e 2}-(2\ln 2-1)$$

$$\Rightarrow\frac{\left(2^2-\sqrt{2}\right)}{\log_e 2}-2\ln 2+1$$

$\therefore\ \alpha=2^2-\sqrt{2},\ \beta=-2,\ \gamma=1$

$\Rightarrow(\alpha+\beta+2\gamma)^2$

$\Rightarrow(2^2-\sqrt{2}-2-2)^2$

$\Rightarrow(\sqrt{2})^2=2$

7. A ray of light through (2,1) is reflected at a point P on the y-axis and then passes through the point (5, 3). If this reflected ray is the directrix of an ellipse with eccentricity $\frac{1}{3}$ and the distance of the nearer focus from this directrix is $\frac{8}{\sqrt{53}}$, then the equation of the other directrix can be:

(1) $11x + 7y + 8 = 0$ or $11x + 7y - 15 = 0$

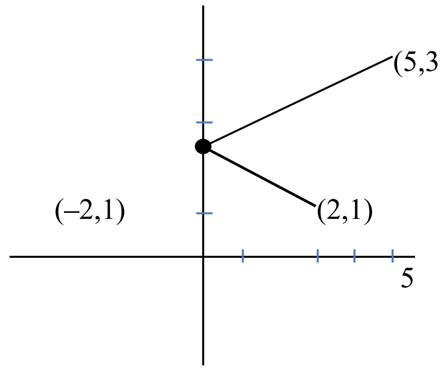
(2) $11x - 7y - 8 = 0$ or $11x + 7y + 15 = 0$

(3) $2x - 7y + 29 = 0$ or $2x - 7y - 7 = 0$

(4) $2x - 7y - 39 = 0$ or $2x - 7y - 7 = 0$

Official Ans. by NTA (3)

Sol.



Equation of reflected Ray

$y-1=\frac{2}{7}(x+2)$

$7y-7=2x+4$

$2x-7y+11=0$

Let the equation of other directrix is

$2x - 7y + \lambda$

Distance of directrix from Focub

$\frac{a}{e}-ae=\frac{8}{\sqrt{53}}$

$3a-\frac{a}{3}=\frac{8}{\sqrt{53}}$ or $a=\frac{3}{\sqrt{53}}$

Distance from other focus $\frac{a}{e}+ae$

$3a+\frac{a}{3}=\frac{10a}{3}=\frac{10}{3}\times\frac{3}{\sqrt{53}}=\frac{10}{\sqrt{53}}$

Distance between two directrix $=\frac{2a}{e}$

$=2\times3\times\frac{3}{\sqrt{53}}=\frac{18}{\sqrt{53}}$

$\left|\frac{\lambda-11}{\sqrt{53}}\right|=\frac{18}{\sqrt{53}}$

$\lambda - 11 = 18$ or -18

$\lambda = 29$ or -7

$2x - 7y - 7 = 0$ or $2x - 7y + 29 = 0$

8. If the coefficients of x^7 in $\left(x^2+\frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(x-\frac{1}{bx^2}\right)^{11}$, $b\ne0$, are equal, then the value of b is equal to:

(1) 2 (2) –1 (3) 1 (4) –2

Official Ans. by NTA (3)

Sol. Coefficient of x^7 in $\left(x^2+\frac{1}{bx}\right)^{11}$

$^{11}C_r(x^2)^{11-r}.\left(\frac{1}{bx}\right)^r$

$^{11}C_r x^{22-3r}.\frac{1}{b^r}$

$22-3r=7$

$r=5$

$\therefore\ ^{11}C_5.\frac{1}{b^5}.x^7$

Coefficient of x^{-7} in $\left(x-\frac{b}{bx^2}\right)^{11}$

$^{11}C_r(x)^{11-r}.\left(-\frac{1}{bx^2}\right)^r$

$^{11}C_r x^{11-3r}.\frac{(-1)^r}{b^r}$

$11-3r=-7\ \therefore r=6$

$^{11}C_6.\frac{1}{b^6}x^{-7}$

$^{11}C_5.\frac{1}{b^5}={}^{11}C_6.\frac{1}{b^6}$

Since $b\neq 0\ \therefore b=1$

9. The compound statement $(P\vee Q)\wedge(\sim P)\Rightarrow Q$ is equivalent to:

(1) $P\vee Q$ (2) $P\wedge\sim Q$

(3) $\sim(P\Rightarrow Q)$ (4) $\sim(P\Rightarrow Q)\Leftrightarrow P\wedge\sim Q$

Official Ans. by NTA (4)

Sol. Using Truth Table

P	Q	$P\vee Q$	$\sim P$	$(P\vee Q)\wedge P$	$(P\vee Q)\wedge\sim P\to Q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

P	Q	$\sim Q$	$P\wedge\sim Q$	$P\to Q$	$\sim(P\to Q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	F	T	F

$\sim(P\to Q)$	$P\wedge\sim Q$	$\sim(P\to Q)\Leftrightarrow P\wedge\sim Q$
F	F	T
T	T	T
F	F	T
F	F	T

10. If $\sin\theta+\cos\theta=\frac{1}{2}$, then

$16(\sin(2\theta)+\cos(4\theta)+\sin(6\theta))$ is equal to:

(1) 23 (2) –27 (3) –23 (4) 27

Official Ans. by NTA (3)

Sol. $\sin\theta+\cos\theta=\frac{1}{2}$

$\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=\frac{1}{4}$

$\sin 2\theta=-\frac{3}{4}$

Now :

$\cos 4\theta=1-2\sin^2 2\theta$

$=1-2\left(-\frac{3}{4}\right)^2$

$=1-2\times\frac{9}{16}=-\frac{1}{8}$

$\sin 6\theta=3\sin 2\theta-4\sin^3 2\theta$

$=(3-4\sin^2 2\theta).\sin 2\theta$

$=\left[3-4\left(\frac{9}{16}\right)\right].\left(-\frac{3}{4}\right)$

$\Rightarrow\left[\frac{3}{4}\right]\times\left(-\frac{3}{4}\right)=-\frac{9}{16}$

$16[\sin 2\theta+\cos 4\theta+\sin 6\theta]$

$16\left(-\frac{3}{4}-\frac{1}{8}-\frac{9}{16}\right)=-23$

11. Let $A=\begin{bmatrix}1 & 2\\ -1 & 4\end{bmatrix}$. If $A^{-1}=\alpha I+\beta A$, $\alpha,\beta\in\mathbf{R}$, I is a 2×2 identity matrix, then $4(\alpha-\beta)$ is equal to :

(1) 5 (2) $\frac{8}{3}$ (3) 2 (4) 4

Official Ans. by NTA (4)

Sol. $A=\begin{bmatrix}1 & 2\\ -1 & 4\end{bmatrix}$, $|A|=6$

$A^{-1}=\frac{\text{adj}A}{|A|}=\frac{1}{6}\begin{bmatrix}4 & -2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}\frac{2}{3} & -\frac{1}{3}\\ \frac{1}{6} & \frac{1}{6}\end{bmatrix}$

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} \beta & 2\beta \\ -\beta & 4\beta \end{bmatrix}$$

$$\left.\begin{array}{l} \alpha+\beta=\frac{2}{3} \\ \beta=-\frac{1}{6} \end{array}\right\} \Rightarrow \alpha=\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$$

$4(\alpha - \beta) = 4(1) = 4$

12. Let $f:\left(-\frac{\pi}{4},\frac{\pi}{4}\right) \to \mathbf{R}$ be defined as

$$f(x)=\begin{cases} (1+|\sin x|)^{\frac{3a}{|\sin x|}} &, -\frac{\pi}{4}<x<0 \\ b &, x=0 \\ e^{\cot 4x/\cot 2x} &, 0<x<\frac{\pi}{4} \end{cases}$$

If f is continuous at x = 0, then the value of $6a + b^2$ is equal to:

(1) $1 - e$ (2) $e - 1$ (3) $1 + e$ (4) e

Official Ans. by NTA (3)

Sol. $\lim\limits_{x\to 0} f(x) = b$

$$\lim_{x\to 0^+} xe^{\frac{\cot 4x}{\cot 2x}} = e^{\frac{1}{2}} = b$$

$$\lim_{x\to 0^-} (1+|\sin x|)^{\frac{3a}{|\sin x|}} = e^{3a} = e^{\frac{1}{2}}$$

$$\lim_{x\to 0^-} (1+|\sin x|)^{\frac{3a}{|\sin x|}} = e^{3a} = e^{\frac{1}{2}}$$

$a = \frac{1}{6} \Rightarrow 6a = 1$

$(6a + b^2) = (1 + e)$

13. Let y = y(x) be solution of the differential equation $\log_e\left(\frac{dy}{dx}\right) = 3x + 4y$, with $y(0) = 0$.

If $y\left(-\frac{2}{3}\log_e 2\right) = \alpha \log_e 2$, then the value of α is equal to:

(1) $-\frac{1}{4}$ (2) $\frac{1}{4}$ (3) 2 (4) $-\frac{1}{2}$

Official Ans. by NTA (1)

Sol. $\frac{dy}{dx} = e^{3x}.e^{4y} \Rightarrow \int e^{-4y}dy = \int e^{3x}dx$

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C \Rightarrow -\frac{1}{4} - \frac{1}{3} = C \Rightarrow C = -\frac{7}{12}$$

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12} \Rightarrow e^{-4y} = \frac{4e^{3x}-7}{-3}$$

$$e^{4y} = \frac{3}{7-4e^{3x}} \Rightarrow 4y = \ell n\left(\frac{3}{7-4e^{3x}}\right)$$

$4y = \ell n\left(\frac{3}{6}\right)$ when $x = -\frac{2}{3}\ell n 2$

$y = \frac{1}{4}\ell n\left(\frac{1}{2}\right) = -\frac{1}{4}\ell n 2$

14. Let the plane passing through the point (–1, 0, –2) and perpendicular to each of the planes $2x + y - z = 2$ and $x - y - z = 3$ be $ax + by + cz + 8 = 0$. Then the value of $a + b + c$ is equal to:

(1) 3 (2) 8 (3) 5 (4) 4

Official Ans. by NTA (4)

Sol. Normal of req. plane $\left(2\hat{i}+\hat{j}-\hat{k}\right)\times\left(\hat{i}-\hat{j}-\hat{k}\right)$

$= -2\hat{i}+\hat{j}-3\hat{k}$

Equation of plane

$-2(x + 1) + 1(y - 0) - 3(z + 2) = 0$

$-2x + y - 3z - 8 = 0$

$2x - y + 3z + 8 = 0$

$a + b + c = 4$

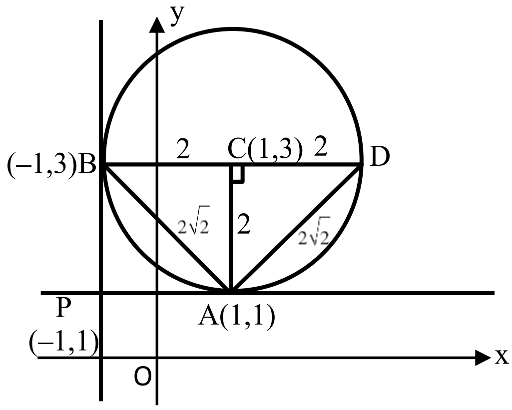
15. Two tangents are drawn from the point P(–1, 1) to the circle $x^2 + y^2 - 2x - 6y + 6 = 0$. If these tangents touch the circle at points A and B, and if D is a point on the circle such that length of the segments AB and AD are equal, then the area of the triangle ABD is equal to:

(1) 2 (2) $(3\sqrt{2}+2)$

(3) 4 (4) $3(\sqrt{2}-1)$

Official Ans. by NTA (3)

Sol.



$$\Delta ABD = \frac{1}{2}\times 2\times 4 = 4$$

16. Let f : **R** □ **R** be a function such that f(2) = 4 and $f'(2)=1$. Then, the value of $\lim\limits_{x\to 2}\frac{x^2 f(2)-4f(x)}{x-2}$ is equal to :

(1) 4 (2) 8 (3) 16 (4) 12

Official Ans. by NTA (4)

Sol. Apply L'Hopital Rule

$$\lim_{x\to 2}\left(\frac{2xf(2)-4f'(x)}{1}\right)$$

$$=\frac{4(4)-4}{1}=12$$

17. Let P and Q be two distinct points on a circle which has center at C(2, 3) and which passes through origin O. If OC is perpendicular to both the line segments CP and CQ, then the set {P, Q} is equal to

(1) $\{(4,0),(0,6)\}$

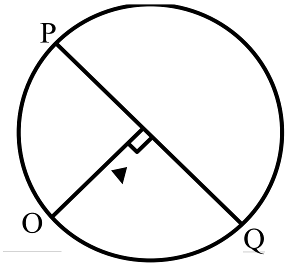
(2) $\left\{\left(2+2\sqrt{2},3-\sqrt{5}\right),\left(2-2\sqrt{2},3+\sqrt{5}\right)\right\}$

(3) $\left\{\left(2+2\sqrt{2},3+\sqrt{5}\right),\left(2-2\sqrt{2},3-\sqrt{5}\right)\right\}$

(4) $\{(-1,5),(5,1)\}$

Official Ans. by NTA (4)

Sol.



$$\tan\theta = -\frac{2}{3}$$

Using symmetric from of line

P,Q : $\left(2\pm\sqrt{13}\cos\theta, 3\pm\sqrt{13}\sin\theta\right)$

$$\left(2\pm\sqrt{13}.\left(-\frac{3}{\sqrt{13}}\right),3\pm\sqrt{13}\left(\frac{2}{\sqrt{13}}\right)\right)$$

(−1, 5) & (5, 1)

18. Let α, β be two roots of the equation $x^2+(20)^{1/4}x+(5)^{1/2}=0$. Then $\alpha^8+\beta^8$ is equal to

(1) 10 (2) 100 (3) 50 (4) 160

Official Ans. by NTA (3)

Sol. $\left(x^2+\sqrt{5}\right)^2=\sqrt{20}\,x^2$

$x^4=-5 \Rightarrow x^8 = 25$

$\alpha^8+\beta^8 = 50$

19. The probability that a randomly selected 2-digit number belongs to the set $\{n \in N : (2^n - 2)$ is a multiple of 3} is equal to

(1) $\frac{1}{6}$ (2) $\frac{2}{3}$ (3) $\frac{1}{2}$ (4) $\frac{1}{3}$

Official Ans. by NTA (3)

Sol. Total number of cases = $^{90}C_1 = 90$

Now, $2^n - 2 = (3-1)^n - 2$

$$^nC_0 3^n - {}^nC_1 3^{n-1} + ... + (-1)^{n-1}.{}^nC_{n-1}3 + (-1)^n.{}^nC_n - 2$$

$$3\left(3^{n-1} - n3^{n-2} + ... + (-1)^{n-1}.n\right) + (-1)^n - 2$$

$(2^n - 2)$ is multiply of 3 only when n is odd

Req. Probability $=\frac{45}{90}=\frac{1}{2}$

20. Let

$A = \{(x,y)\in \mathbf{R}\times\mathbf{R} \mid 2x^2+2y^2-2x-2y=1\}$,

$B = \{(x,y)\in \mathbf{R}\times\mathbf{R} \mid 4x^2+4y^2-16y+7=0\}$ and

$C = \{(x,y)\in \mathbf{R}\times\mathbf{R} \mid x^2+y^2-4x-2y+5\le r^2\}$.

Then the minimum value of $|r|$ such that $A\cup B\subseteq C$ is equal to

(1) $\frac{3+\sqrt{10}}{2}$ (2) $\frac{2+\sqrt{10}}{2}$

(3) $\frac{3+2\sqrt{5}}{2}$ (4) $1+\sqrt{5}$

Official Ans. by NTA (3)

Sol. $S_1 : x^2 + y^2 - x - y - \frac{1}{2} = 0$; $C_1\left(\frac{1}{2},\frac{1}{2}\right)$

$$r_1 = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}} = 1$$

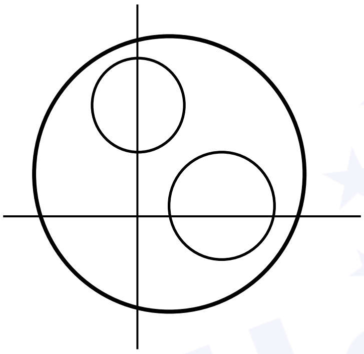
$S_2 : x^2 + y^2 - 4y + \frac{7}{4} = 0$; $C_2 : (0,2)$

$$r_2 = \sqrt{4-\frac{7}{4}} = \frac{3}{2}$$

$S_3 : x^2 + y^2 - 4x - 2y + 5 - r^2 = 0$

$C_3 : (2,1)$

$$r_3 = \sqrt{4+1-5+r^2} = |r|$$



$$C_1C_3 = \sqrt{\frac{5}{2}}$$

$$\sqrt{\frac{5}{2}} \le |r-1| \Rightarrow \left.\begin{matrix} r \le 1+\sqrt{\frac{5}{2}} \\ r \ge \frac{3}{2}+\sqrt{5} \end{matrix}\right\}$$

$$C_2C_3 = \sqrt{5} \le \left|r-\frac{3}{2}\right|$$

$$\left.\begin{matrix} r-\frac{3}{2} \ge \sqrt{5} \\ r-\frac{3}{2} \le -\sqrt{5} \end{matrix}\right\}$$

SECTION-B

1. For real numbers α and β, consider the following system of linear equations :

$x + y - z = 2$, $x + 2y + \alpha z = 1$, $2x - y + z = \beta$.

If the system has infinite solutions, then $\alpha + \beta$ is equal to ________

Official Ans. by NTA (5)

Sol. For infinite solutions

$$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & \alpha \\ 2 & -1 & 1 \end{vmatrix} = 0$$

$$\Delta = \begin{vmatrix} 3 & 0 & 0 \\ 1 & 2 & \alpha \\ 2 & -1 & 1 \end{vmatrix} = 0$$

$$\Delta = 3(2+\alpha) = 0$$

$$\Rightarrow \alpha = -2$$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \\ 2 & \beta & 1 \end{vmatrix} = 0$$

$$1(1+2\beta) - 2(1+4) - (\beta-2) = 0$$

$$\beta - 7 = 0$$

$$\beta = 7$$

$\therefore \alpha + \beta = 5$ Ans.

2. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b}$ and $\vec{c} = \hat{j} - \hat{k}$ be three vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 1$. If the length of projection vector of the vector $\vec{b}$ on the vector $\vec{a} \times \vec{c}$ is l, then the value of $3l^2$ is equal to ________.

Official Ans. by NTA (2)

Sol. $\vec{a} \times \vec{b} = c$

Take Dot with $\vec{c}$

$$(\vec{a} \times \vec{b}).\vec{c} = |\vec{c}|^2 = 2$$

Projection of $\vec{b}$ or $\vec{a} \times \vec{c} = \ell$

$$\frac{|\vec{b}.(\vec{a} \times \vec{c})|}{|\vec{a} \times \vec{c}|} = \ell$$

$$\therefore \ell = \frac{2}{\sqrt{6}} \Rightarrow \ell^2 = \frac{4}{6}$$

$$3\ell^2 = 2$$

3. If $\log_3 2, \log_3(2^x-5), \log_3\left(2^x-\frac{7}{2}\right)$ are in an arithmetic progression, then the value of x is equal to ________.

Official Ans. by NTA (3)

Sol. $2\log_3(2^x-5)=\log_3 2+\log_3\left(2^x-\frac{7}{2}\right)$

Let $2^x=t$

$\log_3(t-5)^2=\log_3 2\left(t-\frac{7}{2}\right)$

$(t-5)^2=2t-7$

$t^2-12t+32=0$

$(t-4)(t-8)=0$

$\Rightarrow 2^x=4$ or $2^x=8$

X = 2 (Rejected)

Or x = 3

4. Let the domain of the function

$f(x)=\log_4\left(\log_5\left(\log_3\left(18x-x^2-77\right)\right)\right)$ be (a, b).

Then the value of the integral

$\int_a^b \frac{\sin^3 x}{(\sin^3 x+\sin^3(a+b-x))}dx$ is equal to ________.

Official Ans. by NTA (1)

Sol. For domain

$\log_5\left(\log_3\left(18x-x^2-77\right)\right)>0$

$\log_3\left(18x-x^2-77\right)>1$

$18x-x^2-77>3$

$x^2-18x+80<0$

$x\in(8,10)$

$\Rightarrow a=8$ and $b=10$

$I=\int_a^b \frac{\sin^3 x}{\sin^3 x+\sin^3(a+b-x)}dx$

$I=\int_a^b \frac{\sin^3(a+b-x)}{\sin^3 x+\sin^3(a+b-x)}$

$2I=(b-a)\Rightarrow I=\frac{b-a}{2}\quad(\because a=8 \text{ and } b=10)$

$I=\frac{10-8}{2}=1$

5. Let

$$f(x)=\begin{vmatrix}\sin^2 x & -2+\cos^2 x & \cos 2x\\ 2+\sin^2 x & \cos^2 x & \cos 2x\\ \sin^2 x & \cos^2 x & 1+\cos 2x\end{vmatrix}, x\in[0,\pi]$$

Then the maximum value of f(x) is equal to ________.

Official Ans. by NTA (6)

Sol. $$\begin{vmatrix}-2 & -2 & 0\\ 2 & 0 & -1\\ \sin^2 x & \cos^2 x & 1+\cos 2x\end{vmatrix}\begin{pmatrix}R_1\to R_1-R_2\\ \& R_2\to R_2-R_3\end{pmatrix}$$

$-2(\cos^2 x)+2(2+2\cos 2x+\sin^2 x)$

$4+4\cos 2x-2(\cos^2 x-\sin^2 x)$

$f(x)=4+\underbrace{2\cos 2x}_{\max=1}$

$f(x)_{\max}=4+2=6$

6. Let F : [3, 5] → **R** be a twice differentiable function on (3, 5) such that

$F(x)=e^{-x}\int_3^x(3t^2+2t+4F'(t))\,dt.$

If $F'(4)=\frac{\alpha e^{\beta}-224}{(e^{\beta}-4)^2}$, then $\alpha+\beta$ is equal to ________.

Official Ans. by NTA (16)

Sol. $F(3)=0$

$e^x F(x)=\int_3^x\left(3t^2+2t+4F'(t)\right)dt$

$e^x F(x)+e^x F'(x)=3x^2+2x+4F'(x)$

$(e^x-4)\frac{dy}{dx}+e^x y=(3x^2+2x)$

$\frac{dy}{dx}+\frac{e^x}{(e^x-4)}y=\frac{(3x^2+2x)}{(e^x-4)}$

$ye^{\int\frac{e^x}{(e^x-4)}dx}=\int\frac{(3x^2+2x)}{(e^x-4)}e^{\int\frac{e^x}{e^x-4}dx}dx$

$y.(e^x-4)=\int(3x^2+2x)dx+c$

$$y(e^x-4)=x^3+x^2+c$$

Put x = 3 ⇒ c = –36

$$F(x)=\frac{(x^3+x^2-36)}{(e^x-4)}$$

$$F'(x)=\frac{(3x^2+2x)(e^x-4)-(x^3+x^2-36)e^x}{(e^x-4)^2}$$

Now put value of x = 4 we will get α = 12 & β = 4

7. Let a plane P pass through the point (3, 7, –7) and contain the line, $\frac{x-2}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$. If distance of the plane P from the origin is d, then d^2 is equal to __________.

Official Ans. by NTA (3)

Sol. $\overrightarrow{BA}=(\hat{i}+4\hat{j}-5\hat{k})$

$\overrightarrow{BA}=(\hat{i}+4\hat{j}-5\hat{k})$

$$\overrightarrow{BA}\times\vec{\ell}=\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 2 & 1\\ 1 & 4 & -5\end{vmatrix}$$

$$a\hat{i}+b\hat{j}+c\hat{k}=-14\hat{i}-\hat{j}(14)+\hat{k}(-14)$$

a = 1, b = 1, c = 1

Plane is (x – 2) + (y – 3) + (z + 2) = 0

x + y + z –3 = 0

$d=\sqrt{3}\Rightarrow d^2=3$

8. Let S = {1, 2, 3, 4, 5, 6, 7}. Then the number of possible functions f : S → S such that f(m·n) = f(m)·f(n) for every m, n ∈ S and m · n ∈ S is equal to______.

Official Ans. by NTA (490)

Sol. F(mn) = f(m). f(n)

Put m = 1 **f(n) = f(1).** f(n) ⇒ f(1) = 1

Put m = n = 2

$$f(4)=f(2).f(2)\begin{cases}f(2)=1\Rightarrow f(4)=1\\ \text{or}\\ f(2)=2\Rightarrow f(4)=4\end{cases}$$

Put m = 2, n = 3

$$f(6)=f(2).f(3)\begin{cases}\text{when } f(2)=1\\ f(3)=1 \text{ to } 7\\ \\ f(2)=2\\ f(3)=1 \text{ or } 2 \text{ or } 3\end{cases}$$

f(5), f(7) can take any value

Total = (1 × 1 × 7 × 1 × 7 × 1 × 7)

+ (1 × 1 × 3 × 1 × 7 × 1 × 7)

= 490

9. If y = y(x), $y\in\left[0,\frac{\pi}{2}\right)$ is the solution of the differential equation

$\sec y\frac{dy}{dx}-\sin(x+y)-\sin(x-y)=0$, with y(0) = 0,

then $5y'\left(\frac{\pi}{2}\right)$ is equal to __________.

Official Ans. by NTA (2)

Sol. $\sec y\frac{dy}{dx}=2\sin x\cos y$

$\sec^2 y\,dy=2\sin x\,dx$

$\tan y=-2\cos x+c$

c = 2

tany = –2cosx + 2 ⇒ at x = $\frac{\pi}{2}$

tan y = 2

$\sec^2 y\frac{dy}{dx}=2\sin x$

$5\frac{dy}{dx}=2$

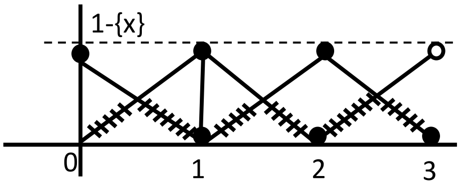
10. Let f : [0, 3] → **R** be defined by

f(x) = min {x – [x], 1 + [x] – x}

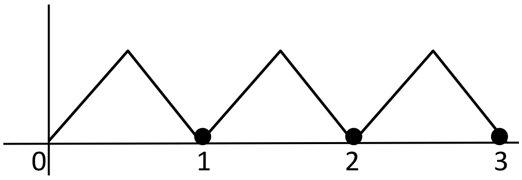
where [x] is the greatest integer less than or equal to x. Let P denote the set containing all x ∈ [0, 3] where f is discontinuous, and Q denote the set containing all x ∈ (0, 3) where f is not differentiable. Then the sum of number of elements in P and Q is equal to __________.

Official Ans. by NTA (5)

Sol.



$1 - \{x\} = 1 - x;\ 0 \leq x < 1$



Non differentiable at

$x = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$