SECTION-A

1. Let the solution curve of the differential equation

$$xdy = \left(\sqrt{x^2+y^2}+y\right)dx, \ x > 0,$$ intersect the line

x = 1 at y = 0 and the line x = 2 at y = α . Then the value of α is :

(A) $\dfrac{1}{2}$ (B) $\dfrac{3}{2}$ (C) $-\dfrac{3}{2}$ (D) $\dfrac{5}{2}$

Official Ans. by NTA (B)

☐ **Ans. (B)**

Sol. $xdy = \left(\sqrt{x^2+y^2}+y\right)dx$

$xdy - ydx = \sqrt{x^2+y^2}\,dx$

$\dfrac{xdy-ydx}{x^2} = \sqrt{1+\dfrac{y^2}{x^2}}\cdot\dfrac{dx}{x}$

$\dfrac{d(y/x)}{\sqrt{1+\left(\dfrac{y}{x}\right)^2}} = \dfrac{dx}{x}$

$\ln\left(\dfrac{y}{x}+\sqrt{\left(\dfrac{y}{x}\right)^2+1}\right) = \ln x + R$

$\dfrac{y\ \sqrt{\ \ +x^2}}{x} = cx$

$y + \sqrt{y^2+x^2} = cx^2$

$x = 1, y = 0 \Rightarrow 0+1 = C \Rightarrow C = 1$

Curve is $y + \sqrt{x^2+y^2} = x^2$

$x = 2, y = \alpha$

$2 + \sqrt{4+\alpha^2} = 4$

$4 + \alpha^2 = 16 + \alpha^2 = 8\alpha$

$\alpha = \dfrac{3}{2}$

2. Considering only the principal values of the inverse trigonometric functions, the domain of the function $f(x) = \cos^{-1}\left(\dfrac{x^2-4x+2}{x^2+3}\right)$ is :

(A) $\left(-\infty,\dfrac{1}{4}\right]$ (B) $\left[-\dfrac{1}{4},\infty\right)$

(C) $\left(-\dfrac{1}{3},\infty\right)$ (D) $\left(-\infty,\dfrac{1}{3}\right]$

Official Ans. by NTA (B)

☐ **Ans. (B)**

Sol.

$\left|\dfrac{x^2+4x+2}{x^2+3}\right| \le 1$

$\Leftrightarrow \left(x^2-4x+2\right)^2 \le \left(x^2+3\right)^2$

$\Leftrightarrow \left(x^2-4x+2\right)^2 - \left(x^2+3\right)^2 \le 0$

$\Leftrightarrow \left(2x^2-4x+5\right)\left(-4x-1\right) \le 0$

$\Leftrightarrow -4x-1 \le 0 \to x \ge -\dfrac{1}{4}$

3. Let the vectors $\vec{a} = (1+t)\hat{i}+(1-t)\hat{j}+\hat{k},$

$\vec{b} = (1-t)\hat{i}+(1+t)\hat{j}+2\hat{k}$ and $\vec{c} = t\hat{i}-t\hat{j}+\hat{k}, \ t \in R$

be such that for $\alpha,\beta,\gamma \in R, \ \alpha\vec{a}+\beta\vec{b}+\gamma\vec{c} = \vec{0}$

$\Rightarrow \alpha = \beta = \gamma = 0.$ Then, the set of all values of t is :

(A) a non-empty finite set

(B) equal to N

(C) equal to R – {0}

(D) equal to R

Official Ans. by NTA (C)

Ans. (C)

Sol. By its given condition

: $\vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors

$$\Rightarrow \left[\,\overline{a}\,\overline{b}\,\overline{c}\,\right] \neq 0 \qquad \text{..(i)}$$

Now, $\left[\,\overline{a}\,\overline{b}\,\overline{c}\,\right]$

$$= \begin{vmatrix} 1+t & 1-t & 1 \\ 1-t & 1+t & 2 \\ & & 1 \end{vmatrix}$$

$C_2 \rightarrow C_1 + C_2$

$$\begin{vmatrix} 1+t & 2 & 1 \\ 1-t & 2 & 2 \\ t & 0 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1+t & 1 & 1 \\ 1-t & 1 & 2 \\ t & 0 & 1 \end{vmatrix}$$

$$= 2\left[(1+t) - (1-t) + t\right]$$

$$= 2[3t] = 6t$$

$$\left[\,\overline{a}\,\overline{b}\,\overline{c}\,\right] \neq 0 \Rightarrow t \neq 0$$

4. Considering the principal values of the inverse trigonometric functions, the sum of all the solutions of the equation $\cos^{-1}(x) - 2\sin^{-1}(x) = \cos^{-1}(2x)$ is equal to :

(A) 0   (B) 1   (C) $\dfrac{1}{2}$ (D) $-\dfrac{1}{2}$

Official Ans. by NTA (A)

    Ans. (A)

Sol. $\cos^{-1}x = 2\sin^{-1}x = \cos^{-1}2x$

$$\cos^{-1}x - 2\left(\frac{\pi}{2} - \cos^{-1}x\right) = \cos^{-1}2x$$

$$\cos^{-1}x - \pi + 2\cos^{-1}x = \cos^{-1}2x$$

$$3\cos^2 x = \pi + \cos^{-1}2x \qquad \ldots(1)$$

$$\cos\left(3\cos^{-1}x\right) = \cos\left(\pi + \cos^{-1}2x\right)$$

$$4x^3 - 3x = -2x$$

$$4x^3 = x \Rightarrow x = 0, \pm\frac{1}{2}$$

All satisfy the original equation

$$\text{sum} = -\frac{1}{2} \text{ to} + \frac{1}{2} = 0$$

5. Let the operations $*, \odot \in \{\wedge, \vee\}$. If $(p * q) \odot (p \odot \sim q)$ is a tautology, then the ordered pair $(*, \odot)$ is :

(A) $(\vee, \wedge)$ (B) $(\vee, \vee)$ (C) $(\wedge, \wedge)$ (D) $(\wedge, \vee)$

Official Ans. by NTA (B)

    Ans. (B)

Sol. Well check each option

For A $\pi = v$ of $0 = \Lambda$
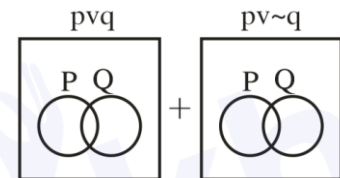
$$(pvq) \wedge (pv \sim q)$$

$$\equiv pv(q \wedge \sim q)$$

$$\equiv pv(c) \equiv p$$

For B : $* = v$, $O = v$

$$(pvq) \vee (pv \sim q) \equiv t \quad \text{using Venn Diagrams}$$

6. Let a vector $\vec{a}$ has a magnitude 9. Let a vector $\vec{b}$ be such that for every $(x, y) \in R \times R - \{(0,0)\}$, the vector $(x\vec{a} + y\vec{b})$ is perpendicular to the vector $(6y\vec{a} - 18x\vec{b})$. Then the value of $\left|\vec{a} \times \vec{b}\right|$ is equal to:

(A) $9\sqrt{3}$ (B) $27\sqrt{3}$ (C) 9   (D) 81

Official Ans. by NTA (B)

    Ans. (B)

Sol. $\left|\vec{a}\right| = 9$ & $\left(x\vec{a} + y\vec{b}\right) \cdot \left(6y\vec{a} - 18x\vec{b}\right) = 0$

$$\Rightarrow 6xy\left|\overline{a}\right|^2 - 18x^2\left(\overline{a} \cdot \overline{b}\right) + 6y^2\left(\overline{a} \cdot \overline{b}\right) - 18xy\left|\overline{b}\right|^2 = 0$$

$$\Rightarrow 6xy\left(\left|\overline{a}\right|^2 - 3\left|\overline{b}\right|^2\right) + \left(\overline{a} \cdot \overline{b}\right)\left(y^2 - 3x^2\right) = 0$$

This should hold $\forall \, x, y \in R \times R$

$$\therefore \left|\overline{a}\right|^2 = 3\left|\overline{b}\right|^2 \, \& \, \left(\overline{a} \cdot \overline{b}\right) = 0$$

Now $\left|\overline{a} \times \overline{b}\right|^2 = \left|\overline{a}\right|^2\left|\overline{b}\right|^2 - \left(\overline{a} \cdot \overline{b}\right)^2$

$$= \left|\overline{a}\right|^2 \cdot \frac{\left|\overline{a}\right|^2}{3}$$

$$\therefore \left|\overline{a} \times \overline{b}\right| = \frac{\left|\overline{a}\right|^2}{\sqrt{3}} = \frac{81}{\sqrt{3}} = 27\sqrt{3}$$

7. For $t \in (0, 2\pi)$, if ABC is an equilateral triangle with vertices A(sin t, –cos t), B(cos t, sin t) and C(a, b) such that its orthocentre lies on a circle with centre $\left(1, \frac{1}{3}\right)$, then $(a^2 – b^2)$ is equal to :

(A) $\frac{8}{3}$ (B) 8

(C) $\frac{77}{9}$ (D) $\frac{80}{9}$

Official Ans. by NTA (B)

Ans. (B)

Sol. $s \equiv \sin t, c \equiv \cos t$

Let orthocentre be (h,k)

Since it if an equilateral triangle hence orthocentre coincides with centroid.

$\therefore a + s + c = 3h, b + s - c = 3k$

$\therefore (3h - a)^2 + (3k - b)^2 = (s + c)^2 + (s - c)^2 = 2(s^2 + c^2) = 2$

$\therefore \left(h - \frac{a}{3}\right)^2 + \left(K - \frac{b}{3}\right)^2 = \frac{2}{9},$

circle centre at $\left(\frac{a}{3}, \frac{b}{3}\right)$

Gives, $\frac{a}{3} = 1, \frac{b}{3} = \frac{1}{3} \Rightarrow a = 3, b = 1$

$\therefore a^2 - b^2 = 8$

8. For $\alpha \in N,$ consider a relation R on N given by $R = \{(x, y) : 3x + \alpha y \text{ is a multiple of } 7\}.$ The relation R is an equivalence relation if and only if :

(A) $\alpha = 14$

(B) α is a multiple of 4

(C) 4 is the remainder when α is divided by 10

(D) 4 is the remainder when α is divided by 7

Official Ans. by NTA (D)

Ans. (D)

Sol. For R to be reflexive $\Rightarrow x R x$

$\Rightarrow 3x + \alpha x = 7x \Rightarrow (3 + \alpha)x = 7K$

$\Rightarrow 3 + \alpha = 7\lambda \Rightarrow \alpha = 7\lambda - 3 = 7N + 4, K, \lambda, N \in I$

$\therefore$ when α divided by 7, remainder is 4.

R to be symmetric $xRy \Rightarrow yRx$

$3x + \alpha y = 7N_1, 3y + \alpha x = 7N_2$

$\Rightarrow (3 + \alpha)(x + y) = 7(N_1 + N_2) = 7N_3$

Which holds when $3 + \alpha$ is multiple of 7

$\therefore \alpha = 7N + 4$ (as did earlier)

R to be transitive

$xRy \& yRz \Rightarrow xRz.$

$3x + \alpha y = 7N_1 \& 3y + \alpha z = 7N_2$ and

$3x + \alpha z = 7N_3$

$\therefore 3x + 7N_2 - 3y = 7N_3$

$\therefore 7N_1 - \alpha y + 7N_2 - 3y = 7N_3$

$\therefore 7(N_1 + N_2) - (3 + \alpha)y = 7N_3$

$\therefore (3 + \alpha)y = 7N$

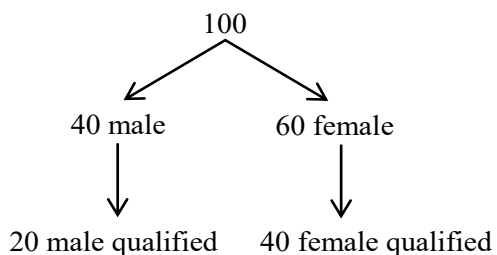Which is true again when $3 + \alpha$ divisible by 7, i.e. when α divided by 7, remainder is 4.

9. Out of 60% female and 40% male candidates appearing in an exam, 60% candidates qualify it. The number of females qualifying the exam is twice the number of males qualifying it. A candidate is randomly chosen from the qualified candidates. The probability, that the chosen candidate is a female, is :

(A) $\frac{3}{4}$ (B) $\frac{11}{16}$

(C) $\frac{23}{32}$ (D) $\frac{13}{16}$

Official Ans. by NTA (A)

Ans. (Bonus)

Sol.

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              100
            /      \
      40 male      60 female
         |            |
  20 male qualified   40 female qualified
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Probability that chosen candidate is female $= \frac{40}{60} = \frac{2}{3}$

10. If $y = y(x)$, $x \in \left(0, \dfrac{\pi}{2}\right)$ be the solution curve of the differential equation

$$\left(\sin^2 2x\right)\dfrac{dy}{dx} + \left(8\sin^2 2x + 2\sin 4x\right)y =$$

$$2e^{-4x}\left(2\sin 2x + \cos 2x\right), \quad \text{with} \quad y\left(\dfrac{\pi}{4}\right) = e^{-\pi},$$

then $y\left(\dfrac{\pi}{6}\right)$ is equal to :

(A) $\dfrac{2}{\sqrt{3}}e^{-2\pi/3}$ (B) $\dfrac{2}{\sqrt{3}}e^{2\pi/3}$

(C) $\dfrac{1}{\sqrt{3}}e^{-2\pi/3}$ (D) $\dfrac{1}{\sqrt{3}}e^{2\pi/3}$

Official Ans. by NTA (A)

    Ans. (A)

Sol. Given differential equation can be re-written as

$$\dfrac{dy}{dx} + \left(8 + 4\cot 2x\right)y = \dfrac{2e^{-4x}}{\sin^2 2x}\left(2\sin x + \cos 2x\right)$$

which is a linear diff. equation.

I.f. $= e^{\int(8 + 4\cot 2x)dx} = e^{8x + 2\mathcal{C}u(\sin 2x)}$

$\quad\quad = e^{8x} \cdot \sin^2 2x$

$\therefore$ solution is

$$y\left(e^{8x} \cdot \sin^2 2x\right) = \int 2e^{4x}\left(2\sin 2x + \cos 2x\right)dx + C$$

$$= e^{4x} \cdot \sin 2x + C$$

Given $y\left(\dfrac{\pi}{4}\right) = e^{-\pi} \Rightarrow C = 0$

$\therefore \; y = \dfrac{e^{-4x}}{\sin 2x}$

$\therefore \; y\left(\dfrac{\pi}{6}\right) = \dfrac{e^{-4 \cdot \frac{\pi}{6}}}{\sin\left(2 \cdot \frac{\pi}{6}\right)} = \dfrac{2}{\sqrt{3}}e^{-\frac{2\pi}{3}}$

11. If the tangents drawn at the points P and Q on the parabola $y^2 = 2x - 3$ intersect at the point R(0, 1), then the orthocentre of the triangle PQR is :

(A) (0, 1)   (B) (2, –1)

(C) (6, 3)   (D) (2, 1)
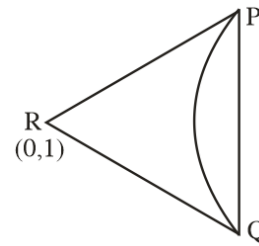
Official Ans. by NTA (B)

    Ans. (B)

Sol. $y^2 = 2x - 3$ ...(i)

Equation of chord of contact

PQ : r = 0

$y \times 1 = (x + 0) - 3$

$y = x - 3$ ...(2)

from (1) and (2)

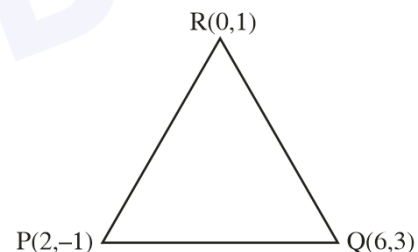$(x \cdot 3)^2 = 2x - 3$

$x^2 - 8x + 12 = 0$

$(x - 2)(x - 6) = 0$

$x = 2 \; \text{or} \; 6$

$y = -1 \; \text{or} \; 3$

$MPQ = \dfrac{1}{4} = 1$

$MQR = \dfrac{2}{6} = \dfrac{1}{3}$

$MPR = \dfrac{2}{6} = \dfrac{1}{3}$

$MPR = \dfrac{2}{-2} = -1$

$MPQ \times MPR = - \Rightarrow PQ \perp PR$

Orthocentre $= P\,(2, -1)$

12. Let C be the centre of the circle $x^2 + y^2 - x + 2y$ $= \dfrac{11}{4}$ and P be a point on the circle. A line passes through the point C, makes an angle of $\dfrac{\pi}{4}$ with the line CP and intersects the circle at the points Q and R. Then the area of the triangle PQR (in unit^2) is :

(A) 2   (B) $2\sqrt{2}$

(C) $8\sin\left(\dfrac{\pi}{8}\right)$ (D) $8\cos\left(\dfrac{\pi}{8}\right)$
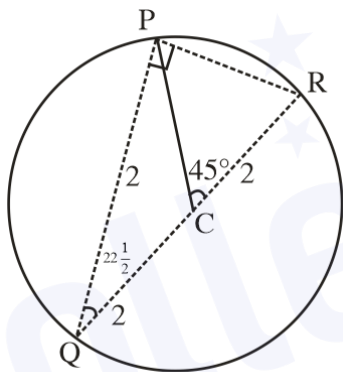
Official Ans. by NTA (B)

☐ **Ans. (B)**

Sol. $x^2 + y^2 - x + 2y = \dfrac{11}{4}$

$\left(x - \dfrac{1}{2}\right)^2 + (y+1)^2 = (2)^2$

Or $\triangle$ PQR

$PR = QK \sin 2 \geq \dfrac{1}{3}$

$= 4 \cdot 6 \sin \dfrac{\pi}{8}$

$PQ = QR \cos 22\dfrac{1}{2}$

$= 4 \cos \dfrac{\pi}{8}$

As $\triangle PQR = \dfrac{1}{2} PR \times PQ$

$= \dfrac{1}{2}\left(4^2 \sin \dfrac{\pi}{6}\right)\left(4\cos\dfrac{\pi}{8}\right)$

$= 4 \sin \dfrac{\pi}{4} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$

13. The remainder when $7^{2022} + 3^{2022}$ is divided by 5 is:

(A) 0   (B) 2   (C) 3   (D) 4

Official Ans. by NTA (C)

    Ans. (C)

Sol. $7^{2022} + 3^{2022}$

$= \left(49\right)^{1011} + \left(9\right)^{1011}$

$= \left(50 - 1\right)^{1011} + \left(10 - 1\right)^{1011}$

$= 5\lambda - 1 + 5K - 1$

$= 5m - 2$

Remainder $= 5 - 2 = 3$

14. Let the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and the matrix

$B_0 = A^{49} + 2A^{98}$. If $B_n = \text{Adj}(B_{n-1})$ for all $n \geq 1$, then $\det(B_4)$ is equal to :

(A) 3^{28} (B) 3^{30} (C) 3^{32} (D) 3^{36}

Official Ans. by NTA (C)

☐ **Ans. (C)**

Sol. $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$a \leftrightarrow R_2$

$-\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$B_0 = A^{49} + 2A^{98}$

$= A + 2I$

$B_n = Adj(B_n - 1)$

$B_4 = Adj(Adj(Adj(AdjB_0)))$

$= |B_0|^{(n-1)^4}$

$= |B_0|^{16}$

$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$= 2(4-0) - 1(0-1)$

$= 9$

$B_4 (9)^{16} = (3)^{32}$

15. Let $S_1 = \left\{ z_1 \in C : |z_1 - 3| = \dfrac{1}{2} \right\}$ and

$S_2 = \left\{ z_2 \in C : |z_2 - |z_2 + 1|| = |z_2 + |z_2 - 1|| \right\}$. Then,

for $z_1 \in S_1$ and $z_2 \in S_2$, the least value of $|z_2 - z_1|$

is :

(A) 0   (B) $\dfrac{1}{2}$ (C) $\dfrac{3}{2}$ (D) $\dfrac{5}{2}$

Official Ans. by NTA (C)

    Ans. (C)

Sol. $|z_2 + |z_2 - 1|| = |z_2 - |z_2 + 1||^2$

$\Rightarrow |z_2 + |z_2 - 1||(\bar{z}_2 + |z_2 - 1|) = (z_2 - |z_2 + 1|)(\bar{z}_2 - (z_2 + 1))$

$\Rightarrow z_2|\bar{z}_2 + 12_2 - 1| - (\bar{z}_2 - |z_2 + 1|) + \bar{z}_2 (|z_2 - 1| + |z_2 + 1|)$
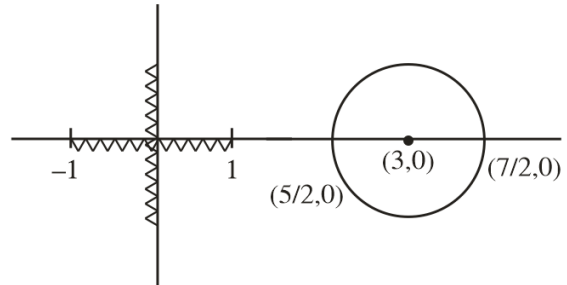
$= |z_2 + 1|^2 = |z_2 - 1|^2$

$\Rightarrow [z_2 + \bar{z}_2)(|z_2 - 1|) + (z_2 + 1|) = 2(z_2 + \bar{z}_2)$

$\Rightarrow (z_2 + \bar{z}_2)(|z_2 - 1| + |z_2 + 1| - 2) = 0$

$\therefore z_2 + \bar{z}_2 = 0$ or $|z_2 - 1| + |z_2 + 1| - 2 = 0$

$\therefore z_2$ lie on imaginary axis. Or on real axis with in $[-1,1]$

Also $|z_1 - 3| = \dfrac{1}{2}$ lie on circle having centre 3 and

radius $\dfrac{1}{2}$.

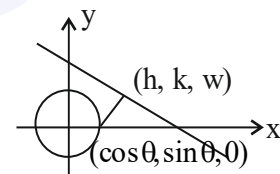Clearly $|z_1 - z_2| \min = \dfrac{5}{2} - 1 = \dfrac{3}{2}$

16. The foot of the perpendicular from a point on the circle $x^2 + y^2 = 1$, $z = 0$ to the plane $2x + 3y + z = 6$ lies on which one of the following curves ?

(A) $(6x + 5y - 12)^2 + 4(3x + 7y - 8)^2 = 1$, $z = 6 - 2x - 3y$

(B) $(5x + 6y - 12)^2 + 4(3x + 5y - 9)^2 = 1$, $z = 6 - 2x - 3y$

(C) $(6x + 5y - 14)^2 + 9(3x + 5y - 7)^2 = 1$, $z = 6 - 2x - 3y$

(D) $(5x + 6y - 14)^2 + 9(3x + 7y - 8)^2 = 1$, $z = 6 - 2x - 3y$

Official Ans. by NTA (B)

    Ans. (B)

Sol.

$\dfrac{h - \cos\theta}{2} = \dfrac{k - \sin\theta}{3} = \dfrac{w - 0}{1}$

$= \dfrac{-1(2\cos\theta + 3\sin\theta - 6)}{14}$

$h = \cos\dfrac{-2(2\cos\theta + 3\sin\theta - 6)}{14}$

$= \dfrac{10\cos\theta - 6\sin\theta + 12}{14}$

$k = \sin\theta - \dfrac{3}{14}(2\cos\theta + 3\sin\theta - 6)$

$k = \dfrac{5\sin\theta - 6\cos\theta + 18}{14}$

Elementary $\sin\theta$ and $\cos\theta$

$(5h + 6k - 12)^2 + 4(3h + 5k - 9)^2 = 1$

17. If the minimum value of $f(x) = \dfrac{5x^2}{2} + \dfrac{\alpha}{x^5}, x > 0$, is 14, then the value of α is equal to :

(A) 32   (B) 64

(C) 128   (D) 256

Official Ans. by NTA (C)

       Ans. (C)

Sol. $\dfrac{x^2}{2} + \dfrac{x^2}{2} + \dfrac{x^2}{2} + \dfrac{x^2}{2} + \dfrac{x^2}{2} + \dfrac{\alpha}{2x^5} + \dfrac{\alpha}{2x^5}$

$\geq 7\left(\dfrac{\alpha^2}{2^7}\right)^{\frac{1}{7}}$

$\dfrac{7 \cdot (\alpha)^{2/7}}{2} = 14$

$(\alpha^2)^{1/7} = 2^2$

$\alpha = \left(2^2\right)^{7/2} = 2^7$

$\alpha = 128$

18. Let α, β and γ be three positive real numbers. Let $f(x) = \alpha x^5 + \beta x^3 + \gamma x, x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be such that $g(f(x)) = x$ for all $x \in \mathbb{R}$. If $a_1, a_2, a_3, ..., a_n$ be in arithmetic progression with mean zero, then the value of $f\left(g\left(\dfrac{1}{n}\sum_{i=1}^{n} f(a_i)\right)\right)$ is equal to :

(A) 0   (B) 3

(C) 9   (D) 27

Official Ans. by NTA (A)

      Ans. (A)

Sol. Consider a case when $\alpha = \beta = 0$ then

$f(x) = \gamma x$

$g(x) = \dfrac{x}{\gamma}$

$\dfrac{1}{n}\sum_{i=1}^{n} f(a_i) \Rightarrow \dfrac{\gamma}{n}\left(a_1 + a_2 + ... + a_n\right)$

$= 0$

$\Rightarrow f(g(0)) \Rightarrow f(0)$

$\Rightarrow 0$

19. Consider the sequence $a_1, a_2, a_3, \ldots\ldots$ such that $a_1 = 1, a_2 = 2$ and $a_{n+2} = \dfrac{2}{a_{n+1}} + a_n$ for $n = 1, 2, 3, \ldots$.

If $\left|\dfrac{a_2}{a_3}\right| \cdot \left|\dfrac{a_3}{a_4}\right| \cdot \left|\dfrac{a_4}{a_5}\right| \ldots\ldots \left|\dfrac{\overline{a_{31}}}{a_{32}}\right| = 2^\alpha \left({}^{61}C_{31}\right)$, then α is equal to :

(A) –30   (B) –31

(C) –60   (D) –61

Official Ans. by NTA (C)

      Ans. (C)

Sol. $a_{n+2}\, a_{n+1} - a_{n+1}.a_n = 2$

Series will satisfy

$a_1 a_2, a_2 a_3, a_3 a_4, a_4 a_5,$

$1.2 \quad 2.2 \quad 2.3 \quad 2.4$

$\dfrac{a_n + \dfrac{1}{a_{n+1}}}{a_{n+2}} = \dfrac{a_{n+2} - \dfrac{1}{a_{n+1}}}{a_{n+2}}$

$= 1 - \dfrac{1}{a_{n+1}a_{n+2}}$

$= 1 - \dfrac{1}{2(r+1)}$

$= \dfrac{2r+1}{2(r+1)}$

Now proof is given by

$= \prod_{r=1}^{30} \dfrac{(2r+1)}{2(r+1)}$

$= \dfrac{(1 \cdot 3 \cdot 5 \cdot ....\cdot 61)}{2^{30} \cdot (2 \cdot 3 \cdot ....\cdot 31)}$

$\Rightarrow \dfrac{(1 \cdot 3 \cdot 5 \cdot ....\cdot 61)}{\lfloor 31 \cdot 2^{30}} \times \dfrac{2^{30} \times \lfloor 30}{2^{30} \times \lfloor 30}$

$= \dfrac{\lfloor 61}{2^{60}\lfloor 31 \cdot \lfloor 30}$

$\alpha = -60$

20. The minimum value of the twice differentiable function $f(x) = \int\limits_{0}^{x} e^{x-t} f'(t)\,dt - (x^2 - x + 1)e^x$, $x \in R$, is :

(A) $-\dfrac{2}{\sqrt{e}}$ (B) $-2\sqrt{e}$

(C) $-\sqrt{e}$ (D) $\dfrac{2}{\sqrt{e}}$

Official Ans. by NTA (A)

Ans. (A)

Sol. $f(x) = e^x \cdot \int\limits_{0}^{x} \dfrac{f'(t)}{e^t}\,dt$

$f'(x) = e^x \cdot \int\limits_{0}^{x} \dfrac{f'(t)}{e^t}\,dt + e^x \cdot \dfrac{f'(x)}{e^x}$

$\qquad\qquad - \left[(2x-1)\cdot e^x + (x^2 - x + 1)\cdot e^x\right]$

$\int\limits_{0}^{x} \dfrac{f'(t)}{e^t}\,dt = x^2 + x$

$\dfrac{f'(x)}{e^x} = 2x + 1$

$f'(x) = (2x+1)\cdot e^x$

$f'(x) = 0 \Rightarrow x = -\dfrac{1}{2}$

$f(x) = (2x+1)\cdot e^x - 2e^x + C$

$\Big|\, f(0) = -1$

$-1 = 1 - 2 + C$

$C = 0$

$f(x) = e^x(2x - 1)$

$f\left(-\dfrac{1}{2}\right) = \dfrac{-2}{\sqrt{e}}$

SECTION-B

1. Let S be the set of all passwords which are six to eight characters long, where each character is either an alphabet from {A, B, C, D, E} or a number from {1, 2, 3, 4, 5} with the repetition of characters allowed. If the number of passwords in S whose at least one character is a number from {1, 2, 3, 4, 5} is $\alpha \times 5^6$, then α is equal to _____.

Official Ans. by NTA (7073)

Ans. (7073)

Sol. Required no. = Total – no character from {1, 2, 3, 4, 5}

$= (10^6 - 5^6) + (10^7 - 5^7) + (10^8 - 5^8)$

$= 10^6(1 + 10 + 100) - 5^6(1 + 5 + 25)$

$= 10^6 \times 111 - 5^6 \times 31$

$= 2^6 \times 5^6 \times 111 - 5^6 \times 31$

$= 5^6(2^6 \times 111 - 31)$

$= 5^6 \times \underset{\alpha}{7073}$

$\therefore \alpha = 7073$

2. Let $P(-2, -1, 1)$ and $Q\left(\dfrac{56}{17}, \dfrac{43}{17}, \dfrac{111}{17}\right)$ be the vertices of the rhombus PRQS. If the direction ratios of the diagonal RS are $\alpha, -1, \beta$, where both α and β are integers of minimum absolute values, then $\alpha^2 + \beta^2$ is equal to _____.

Official Ans. by NTA (450)

Ans. (450)

Sol. $RS \equiv (\alpha, -1, \beta)$

$DR \text{ of } PQ \equiv \left(\dfrac{56}{17} + 2, \dfrac{43}{17} + 1, \dfrac{111}{17} - 1\right)$

$\equiv \left(\dfrac{90}{17}, \dfrac{60}{17}, \dfrac{94}{17}\right)$

$\underline{90} \qquad \underline{60} \qquad \underline{94}$

$90\alpha + 94\beta = 60$

$\beta = \dfrac{60 - 90\alpha}{94}$

$\beta = \dfrac{30(2 - 3\alpha)}{94}$

$\beta = -30\dfrac{(3\alpha - 2)}{94}$

$\beta = \dfrac{-15}{47}(3\alpha - 2)$

$\Rightarrow \dfrac{\beta}{-15} = \dfrac{3\alpha - 2}{47}$

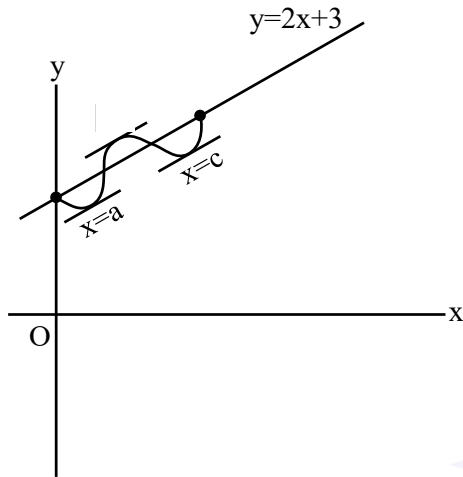$\Rightarrow \beta = -15, \alpha = -15$

$\alpha^2 + \beta^2 = 225 + 225$

$= 450$

3. Let $f:[0,1]\to R$ be a twice differentiable function in $(0, 1)$ such that $f(0) = 3$ and $f(1) = 5$. If the line $y = 2x + 3$ intersects the graph of f at only two distinct points in $(0, 1)$, then the least number of points $x \in (0, 1)$, at which $f''(x) = 0$, is _____.

Official Ans. by NTA (2)

Ans. (2)

Sol.

$f'(a) = f'(b) = f'(c) = 2$

$\Rightarrow f''(x)$ is zero

for atleast $x_1 \in (a, b)$ & $x_2 \in (b, c)$

4. If $\int_0^{\sqrt{3}} \dfrac{15x^3}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}\,dx = \alpha\sqrt{2}+\beta\sqrt{3}$, where α, β are integers, then $\alpha + \beta$ is equal to

Official Ans. by NTA (10)

Ans. (10)

Sol. Put $1 + x^2 = t^2$

$2x\,dx = 2t\,dt$

$X\,dx = t\,dt$

$\therefore \int_1^2 \dfrac{15(t^2-1)t\,dt}{\sqrt{t^2+t^3}}$

$15\int_1^2 \dfrac{t(t^2-1)}{t\sqrt{1+t}}\,dt$

Put $1 + t = u^2$

$dt = 2u\,du$

$15\int_{\sqrt{2}}^{\sqrt{3}} \dfrac{(u^2-1)^2-1}{u} \times 2u\,du$

$30\int_{\sqrt{2}}^{\sqrt{3}} (u^4-2u^2)\,du$

$30\left(\dfrac{u^5}{5}-\dfrac{2u^3}{3}\right)_{\sqrt{2}}^{\sqrt{3}}$

$30\left[\dfrac{1}{5}\left(\sqrt{3}^5-\sqrt{2}^5\right)-\dfrac{2}{3}\left(\sqrt{3}^3-\sqrt{2}^3\right)\right]$

$30\left[\dfrac{1}{5}\left(9\sqrt{3}-4\sqrt{2}\right)-\dfrac{2}{3}\left(3\sqrt{3}-2\sqrt{2}\right)\right]$

$30\left[-\dfrac{1}{5}\times\sqrt{3}+\dfrac{8}{15}\sqrt{2}\right]$

$-6\sqrt{3}+16\sqrt{2}=\alpha\sqrt{2}+\beta\sqrt{3}$

$\alpha = 16, \beta = -6$

$\therefore \alpha + \beta = 10$

5. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix}$ and $B = \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix}$, $\alpha, \beta \in R$. Let α_1 be the value of α which satisfies $(A+B)^2 = A^2 + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and α_2 be the value of α which satisfies $(A+B)^2 = B^2$. Then $|\alpha_1 - \alpha_2|$ is equal to _____.

Official Ans. by NTA (2)

Ans. (2)

Sol. $A + B = \begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix}$

$(A+B)^2 = \begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix}\begin{bmatrix} \beta+1 & 0 \\ 3 & \alpha \end{bmatrix}$

$= \begin{bmatrix} (\beta+1)^2 & 0 \\ 3(\beta+1)+3\alpha & \alpha^2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & \alpha \end{bmatrix}$

$= \begin{bmatrix} -1 & -1-\alpha \\ 2+2\alpha & \alpha^2-2 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & -\alpha+1 \\ 2\alpha+4 & \alpha^2 \end{bmatrix} = \begin{bmatrix} (\beta+1)^2 & 0 \\ 3(\alpha+\beta+1) & \alpha^2 \end{bmatrix}$

$\boxed{\alpha = 1} = \alpha_1$

$B^2 = \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} \beta^2+1 & \beta \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} (\beta+1)^2 & 0 \\ 3(\beta+1)+3\alpha & \alpha^2 \end{bmatrix}$

$\therefore \beta = 0, \alpha = -1 = \alpha_2$

$|\alpha_1 - \alpha_2| = |1-(-1)| = 2$

6. For $p, q \in R$, consider the real valued function $f(x) = (x - p)^2 - q$, $x \in R$ and $q > 0$. Let a_1, a_2, a_3 and a_4 be in an arithmetic progression with mean p and positive common difference. If $|f(a_i)| = 500$ for all i = 1, 2, 3, 4, then the absolute difference between the roots of $f(x) = 0$ is

Official Ans. by NTA (50)

Ans. (50)

Sol. $f(x) = 0 \Rightarrow (x - p)^2 - q = 0$.

Roots are $p + \sqrt{q}$, $p - \sqrt{q}$ absolute difference between roots $2\sqrt{q}$.

Now, $|f(a_i)| = 500$

Let a_1, a_2, a_3, a_4 are $a_1 \ a + d, a + 2d, a + 3d$

$|f(a_4)| = 500$

$|(a_1 - p)^2 - q| = 500$

$\Rightarrow (a_1 - p)^2 - q = 500$

$\Rightarrow \dfrac{9}{4}d^2 - q = 500$ _____(1)

and $|f(a_1)|^2 = |f(a_2)|^2$

$((a_1 - p)^2 - q)^2 = ((a_2 - p)^2 - q)^2$

$\Rightarrow ((a_1 - p)^2 - (a_2 - p)^2)((a_1 - p)^2 - q + (a_2 - p)^2 - q) = 0$

$\Rightarrow \dfrac{9}{4}d^2 - q + \dfrac{d^2}{4} - q = 0$

$2q = \dfrac{10d^2}{4} \Rightarrow q = \dfrac{5d^2}{4}$

$\Rightarrow d^2 = \dfrac{4q}{5}$

From equation (1) $\dfrac{9}{4} \cdot \dfrac{4.q}{5} - q = 500$

$\dfrac{4q}{5} = 500$

$\dfrac{4q}{5} = 500$

and $2\sqrt{q} = 2 \times \dfrac{50}{2} = 50$

7. For the hyperbola $H : x^2 - y^2 = 1$ and the ellipse $E : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, a > b > 0,$ let the

(1) eccentricity of E be reciprocal of the eccentricity of H, and

(2) the line $y = \sqrt{\dfrac{5}{2}}x + K$ be a common tangent of E and H.

Then $4(a^2 + b^2)$ is equal to _____.

Official Ans. by NTA (3)

Ans. (3)

Sol. $e_E = \sqrt{1 - \dfrac{b^2}{a^2}}$, $e_H = \sqrt{2}$

If $\Rightarrow e_E = \dfrac{1}{e_H}$

$\Rightarrow \dfrac{a^2 - b^2}{a^2} = \dfrac{1}{2}$

$2a^2 - 2b^2 = a^2$

$a^2 = 2b^2$

and $y = \sqrt{\dfrac{5}{2}}x + k$ is tangent to ellipse then

$\quad^2 \quad^2 \dfrac{5}{2} \quad^2 \dfrac{3}{2}$

$6b^2 = \dfrac{3}{2} \Rightarrow b^2 = \dfrac{1}{4}$ and $a^2 = \dfrac{1}{2}$

$\therefore 4.(a^2 + b^2) = 3$

8. Let $x_1, x_2, x_3, \ldots, x_{20}$ be in geometric progression with $x_1 = 3$ and the common ration $\dfrac{1}{2}$. A new data is constructed replacing each x_i by $(x_i - i)^2$. If $\bar{x}$ is the mean of new data, then the greatest integer less than or equal to $\bar{x}$ is _____.

Official Ans. by NTA (142)

Ans. (142)

Sol.

$$\sum x_0^1 = \frac{3\left(1-\left(\frac{1}{2}\right)\right)^{20}}{1-\frac{1}{2}} = 6\left(1-\frac{1}{2^{20}}\right)$$

$$= \sum_{i=1}^{20}\left(x_{i-i}\right)^2$$

$$= \sum_{i=1}^{20}\left(x_i\right)^2 + \left(i\right)^2 - 2x_i i$$

Now $$= \sum_{i=1}^{20}\left(x_i\right)^2 = \frac{9\left(1-\left(\frac{1}{4}\right)\right)^{20}}{1-\frac{1}{4}} = 12\left(1-\frac{1}{2^{40}}\right)$$

$$\sum_{i=1}^{20} i^2 = \frac{1}{6} \times 20 \times 21 \times 41 = 2870$$

$$\sum_{i=1}^{20} x_i \, i = s = 3 + 2.3\frac{1}{2} + 3.3\frac{1}{2^2} + 4.3\frac{1}{2^3} + \ldots\ldots \text{AGP}$$

$$= 6\left(2 - \frac{22}{2^{20}}\right)$$

$$\overline{x} = \frac{12 - \frac{12}{2^{40}} + 2870 - 12\left(2 - \frac{22}{2^{20}}\right)}{20}$$

$$\overline{x} = \frac{2858}{20} + \left(\frac{-12}{2^{40}} + \frac{22}{2^{20}}\right) \times \frac{1}{20}$$

$$\left[\overline{x}\right] = 142$$

9. $$\lim_{x \to 0}\left(\frac{(x+2\cos x)^3 + 2(x+2\cos x)^2 + 3\sin(x+2\cos x)}{(x+2)^3 + 2(x+2)^2 + 3\sin(x+2)}\right)^{\frac{100}{x}}$$

is equal to _____.

Official Ans. by NTA (1)

Ans. (1)

Sol.

$$\lim_{x \to 10}\left(\frac{(x+2\cos x)^3 + 2(x+2\cos x)^2 + 3\sin(x+2\cos x)}{2)^3 + 2(x+2)^2 + 3\sin(x+2)}\right)^{x}$$

Form 1^{∞}

$$= e^{\lim_{x \to 0}\left[\left(\frac{(x+2\cos x)^3 + 2(x+2\cos x)^2 + 3\sin(x+2\cos x)}{(x+2)^3 + 2(x+2)^2 + 3\sin(x+2)}\right)-1\right] \times \frac{100}{x}}$$

$$= e^{\lim_{x \to 0}\left[\frac{100}{x}\left(\frac{(x+2\cos x)^3 + 2(x+2\cos x)^2 + 3\sin(x+2\cos x) - \left((x+2)^3 + 2(x+2)^2 + 3\sin(x+2)\right)}{(x+2)^3 + 2(x+2)^2 + 3\sin(x+2)}\right)\right]}$$

$$= e^{\lim_{x \to 0}\frac{100}{x}\left[\left(\frac{(x+2\cos x)^3 + (x+2)^3 + 2(x+2\cos x)^2 - 2(x+2)^2 + 3\sin(x+2\cos x) - 3\sin(x+2)}{8+8+3\sin 2}\right)\right]}$$

$$= e^{\frac{100}{16+3\sin^2}\lim_{x \to 0}\frac{3(x+2\cos x)^2 \times (1+2\sin x) - 3(x+2)^2 - 4(x+2\cos x)}{x(1-2\sin x) - 4(x+2) + 3\cos(x+2\cos x) \times (1-2\sin x) - 3\cos(x+2)}}$$

$$= e^{\frac{100}{16+3\sin 2}\left(\frac{12 - 3(4) + 8 \times 1 - 8 + 3\cos 2 - 3\cos 2}{1}\right)}$$

Using L'H rule.

$$= e^0 = 1$$

10. The sum of all real values of x for which

$$\frac{3x^2 - 9x + 17}{x^2 + 3x + 10} = \frac{5x^2 - 7x + 19}{3x^2 + 5x + 12} \text{ is equal to ___.}$$

Official Ans. by NTA (6)

Ans. (6)

Sol. $$\frac{3x^2 - 9x + 17}{x^2 + 3x + 10} = \frac{5x^2 - 7x + 19}{3x^2 + 5x + 12}$$

$$\frac{x^2 + 3x + 10 + 2x^2 - 12x + 7}{x^2 + 3x + 10} = \frac{3x^2 + 5x + 12 + 2x^2 - 12x + 7}{3x^2 + 5x + 12}$$

$$1 + \frac{2x^2 - 12x + 7}{x^2 + 3x + 10} = 1 + \frac{2x^2 - 12x + 7}{3x^2 + 5x + 12}$$

$$\left(2x^2 - 12x + 7\right)\left(\frac{1}{x^2 + 3x + 10} - \frac{1}{3x^2 + 5x + 12}\right) = 0$$

$$2x^2 - 12x + 7 = 0 \text{ OR } 3x^2 + 5x + 12 = x^2 + 3x + 10$$

$$x = \frac{12 \pm \sqrt{D}}{4} \qquad\qquad 2x^2 + 2x + 2 = 0$$

$$\qquad\qquad\qquad\qquad\qquad x^2 + x + 1 = 0$$

Sum of Roots = 6   No solution.