GS2025
Computer Science & Learning, Information and Data Sciences
Final Answer Key

GS 2025: Solutions for Computer Science (CS) and Learning, Information and Data Sciences (LIDS)

Common Section

1. What is the smallest positive integer that when multiplied with $2^3 3^2 4^3 5^7 6^5$ results in a perfect square?

(a) 6

(b) 10

(c) 15 ✓

(d) 60

(e) 210

2. Assume that in a room there are k persons. We know that $k = 3$ with probability $\frac{1}{4}$, $k = 5$ with probability $\frac{1}{4}$, and $k = 10$ with probability $\frac{1}{2}$. What is the probability that at least two persons in the room have their birthday on the same day of the week? Assume that all days of the week are equally likely.

(a) $\frac{1}{2}(1 + \frac{15\times61}{49^2})$

(b) $1 - \frac{1}{2}(\frac{30}{49} + \frac{360}{49^2})$

(c) $1 - \frac{15}{98}(1 + \frac{12}{49})$ ✓

(d) $1 - \frac{15\times61}{49^2}$

(e) $\frac{4}{7}$

3. How many distinct factors does $2^2 3^3 4^4 5^5$ have? For example, 12 has 6 distinct factors: 1, 2, 3, 4, 6, and 12.

(a) 240

(b) 264 ✓

(c) 324

(d) 360

(e) None of the above

4. There are two bags, one with 10 red balls and 90 blue balls, and another with 90 red balls and 10 blue balls. You pick one of the two bags uniformly at random and draw a random ball from it, and it turns out to be red. You throw that ball away, and draw another from the same bag. Let p be the probability that this ball is also red.

Which of the following is true?

(a) $0 \leq p \leq 0.33$

(b) $0.33 \leq p < 0.50$

(c) $p = 0.5$

(d) $0.50 < p < 0.66$

(e) $0.66 \leq p \leq 1$ ✓

5. There are three boxes. One box contains ten red socks, another contains ten blue socks, and the third contains a mix five red and five blue socks. The three boxes are labelled R (for red), B (for blue), or M (for mixed), but the labels are permuted so that for EACH drawer the label attached to it does NOT match its contents.

How many total socks do you have to look at from the drawers, in order to correctly identify each drawer? (You are not allowed to look inside the drawers. You can only pick socks one at a time and observe its colour.)

(a) One sock. ✓

(b) Two socks.

(c) Three socks.

(d) Four socks.

(e) Six socks.

6. Consider the following matrix:

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ x & 0 & 1 \end{bmatrix}.$$

If all eigenvalues of the matrix are the same, what is the value of x?

(a) 2

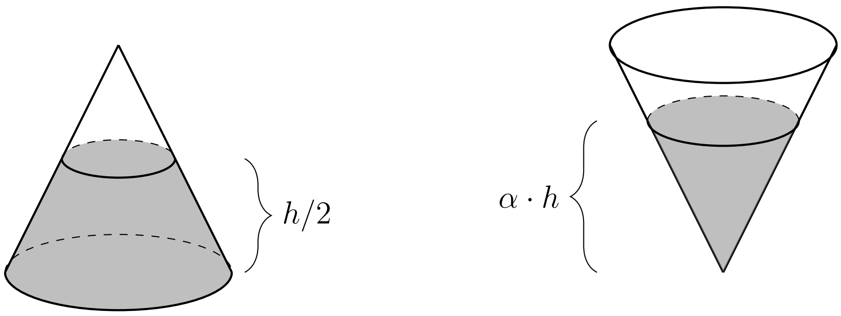
(b) -2

(c) 1

(d) -1 ✓

(e) 0

7. Suppose we had a conical container with some liquid filled in that reaches until half its height when placed with the circular face downwards. If we were to turn the cone upside-down, suppose that the liquid reaches until $\alpha \cdot h$ height when measured from the bottom (the pointy end). (Diagram below not drawn to scale.)



Which of the following is true about α?

(a) $\alpha \leq 0.5$

(b) $0.5 < \alpha < 0.6$

(c) $0.6 < \alpha \leq 0.7$

(d) $0.7 < \alpha \leq 0.8$

(e) $0.8 < \alpha \leq 1$ ✓

8. Consider the collection $\mathcal{C}$ of all possible $n \times n$ matrices with entries in $\{1, \ldots, n\}$ that obey the following property:

- every row of M is a permutation of $\{1, 2, \ldots, n\}$ and every column of M is a permutation of $\{1, 2, \ldots, n\}$.

An example of such a matrix for $n = 3$ is the following.

$$M = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Suppose we choose a matrix X from $\mathcal{C}$ uniformly at random. What is the expected value of $\mathsf{trace}(X)$? Recall that the trace of a matrix is the sum of its diagonal entries.

(a) n

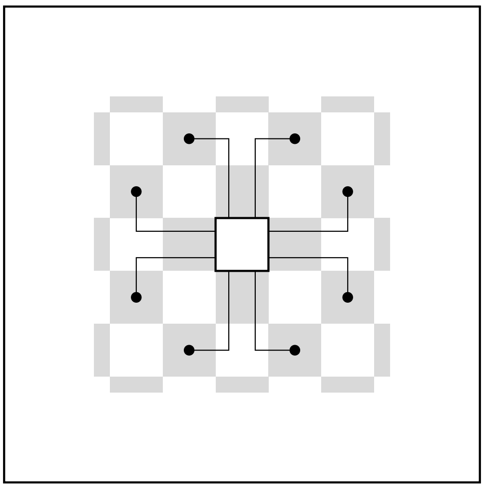
(b) $n^2/2$

(c) $n(n+1)/2$ ✓

(d) $(n+1)^2/2$

(e) n^2

9. What is the largest number of knights that can be placed on a 60×60 chessboard such that no two knights attack each other (two knights attack each other if they are two steps away vertically and one step away horizontally, or vice versa).



The figure above shows a part of a chessboard showing all those squares that a knight placed in the central white square attacks.

(a) 1200
(b) 1500
(c) 1800 ✓
(d) 2100
(e) 2400

10. A clock shows 3:00. After 10^{1024} minutes, what minute does the minute hand point to?

(a) 0
(b) 20
(c) 40 ✓
(d) 50
(e) 10

11. How many distinct ways are there to paint the 6 faces of a cube using six colors (1, 2, 3, 4, 5, 6), with each face painted a different color, such that no two painted cubes are identical (i.e., cannot be transformed into each other by any rotation)?

(a) 30. ✓
(b) 60.
(c) 120.
(d) 360.
(e) 720.

12. Consider a coin which has probability p of coming up heads when tossed. Assume that $0 < p < 1$. A *trial* is performed by repeatedly tossing the coin until the first time it shows a head: the number of tosses made is then called the *value* of the trial. (Thus, if a trial results in tosses "tails", "tails", "heads", then its value is 3.) An *experiment* is a sequence of n independent trials.

Akhil and Bela each perform an experiment, as defined above, with the coin. Akhil defines the value A of his experiment to be the *maximum* of the values of the n trials in his experiment, while Bela defines the value B of her experiment to be the *average* of the value of the n trials in her experiment. Which of the following is correct? (Here, for any random variable X, $E[X]$ denotes the expectation of X.)

(a) $E[A]$ is a strictly increasing function of n while $E[B]$ does not depend upon n. ✓

(b) $E[A]$ is a non-constant function of n, but it is *not* strictly increasing, while $E[B]$ does not depend upon n.

(c) $E[A]$ and $E[B]$ are both strictly increasing functions of n.

(d) $E[A]$ is a non-decreasing function of n while $E[B]$ is a strictly decreasing function of n.

(e) Neither $E[A]$ nor $E[B]$ depends upon n.

13. Let $(X, Y, Z) \in \mathbb{R}^3$ be a point chosen uniformly at random from the unit sphere $S^2 := \{(a, b, c) \in \mathbb{R}^3 | a^2 + b^2 + c^2 = 1\}$ in three dimensions. What is the variance of the random variable Z?

(a) $\frac{1}{2}$

(b) $\frac{1}{3}$ ✓

(c) $\frac{1}{4}$

(d) $\frac{1}{5}$

(e) 1

14. Let $c_1, c_2, \ldots, c_n$ and z be complex numbers such that

$$\frac{1}{z - c_1} + \frac{1}{z - c_2} + \ldots + \frac{1}{z - c_n} = 0.$$

Assume that the numbers $c_1, c_2, \ldots, c_n$ are represented in the complex plane by the vertices of a convex n-gon C. Then

(a) the number z always lies strictly outside C.

(b) the number z always lies inside or on C. ✓

(c) the number z may lie inside or outside C depending on $c_1, c_2, \ldots, c_n$.

(d) if z lies inside C then z must be the centroid, *i.e.*, $z = \frac{1}{n} \sum_{k=1}^{n} c_k$.

(e) None of the above

15. How many subsets of $\{1, 2, \ldots, 2n\}$ do not contain two numbers with sum $2n + 1$?

(a) $3^n + \binom{2n}{2}$

(b) 3^n ✓

(c) $4^n - \binom{2n}{2}$

(d) 2^n

(e) None of the above

Computer Science Section

1. What is the solution to the following recursion?

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{3}\right) + T\left(\frac{n}{6}\right) + O(n),$$
$$T(n) = 5 \quad \forall n < 100.$$

(a) $T(n) = \Theta(n \log n)$ ✓
(b) $T(n) = \Theta(n \log^2 n)$
(c) $T(n) = \Theta(n^2)$
(d) $T(n) = \Theta(n^2 \log n)$
(e) $T(n) = 2^{\Theta(n)}$

2. Let φ be a propositional formula on n variables, with $n \geq 1$. Consider the following statements.

(i) φ is satisfiable
(ii) $\neg\varphi$ is unsatisfiable
(iii) φ is a tautology
(iv) $\neg\varphi$ is a contradiction

Which of the following is ALWAYS TRUE for all such formulas φ?

(a) (i) $\not\Leftrightarrow$ (ii), and (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) ✓
(b) (i) $\not\Leftrightarrow$ (ii), and (ii) $\not\Leftrightarrow$ (iii), and (iii) $\Leftrightarrow$ (iv)
(c) (i) $\not\Leftrightarrow$ (ii), and (iii) $\not\Leftrightarrow$ (iv), and (ii) $\Leftrightarrow$ (iii)
(d) (i) $\not\Leftrightarrow$ (ii), and (ii) $\not\Leftrightarrow$ (iii), and (iii) $\not\Leftrightarrow$ (iv)
(e) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)

3. Consider the following sets.

(i) The set of all natural numbers $\mathbb{N}$.
(ii) The set of all rational numbers $\mathbb{Q}$.
(iii) The set of all functions from $\mathbb{N}$ to $\{1, 2, 3, 4, 5\}$.
(iv) The set of all functions from $\{1, 2, 3, 4, 5\}$ to $\mathbb{N}$.

Which of the above sets are countable?

(a) Only (i)
(b) Only (i) and (ii)
(c) Only (i) and (iv)
(d) Only (i), (ii) and (iii)
(e) Only (i), (ii) and (iv) ✓

4. Consider the following languages:

- L_1 is the set of languages recognised by a deterministic pushdown automaton.
- L_2 is the set of languages recognised by a nondeterministic pushdown automaton.
- L_3 is the set of languages recognised by a nondeterministic Turing machine.
- L_4 is the set of languages recognised by a nondeterministic Turing machine with two tapes.

Choose the correct option below.

(a) $L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq L_4$

(b) $L_1 \subsetneq L_2 \subsetneq L_3 = L_4$ ✓ .

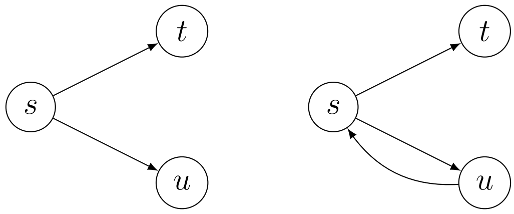
(c) $L_1 \subsetneq L_2 = L_3 = L_4$

(d) $L_1 = L_2 \subsetneq L_3 = L_4$

(e) $L_1 \subseteq L_2 = L_3 \subsetneq L_4$

5. Given a directed graph G and an initial vertex s, we would like to *explore* the graph from s, that is, starting from s see all vertices along a path.

For example, the graph on the left side below cannot be explored from s since there does not exist a path along which we can visit all vertices starting from s while the graph on the right side can be explored from s by following the path $s \to u \to s \to t$.



Which of the following is TRUE?

(a) There exists a polynomial time algorithm to explore the graph from s. ✓

(b) There exists an exponential time algorithm to explore the graph from s but there does not exist any polynomial time algorithm.

(c) There exists an algorithm to explore the graph from s but it is not known if it runs in exponential time.

(d) The decidability of the problem is open.

(e) The halting problem of Turing machine can be reduced to checking if a graph can be explored from s.

6. The complexity class NP corresponds to the class of languages which can be accepted by some nondeterministic Turing machine in polynomial time.

The complexity class coNP consists of all languages whose complement language is in the class NP.

A complexity class $\mathcal{C}$ is said to be *closed under union* if for any two languages L_1 and L_2 in $\mathcal{C}$, we also have that $L_1 \cup L_2 \in \mathcal{C}$. Similarly, we also say that a complexity class $\mathcal{C}$ is *closed under intersection* if for any two languages L_1 and L_2 in $\mathcal{C}$, we also have that $L_1 \cap L_2 \in \mathcal{C}$.

Consider the following statements.

(i) The class NP is closed under union.

(ii) The class coNP is closed under union.

(iii) The class NP is closed under intersection.

(iv) The class coNP is closed under intersection.

Which of the above statements are TRUE?

(a) Only (i) and (iv)

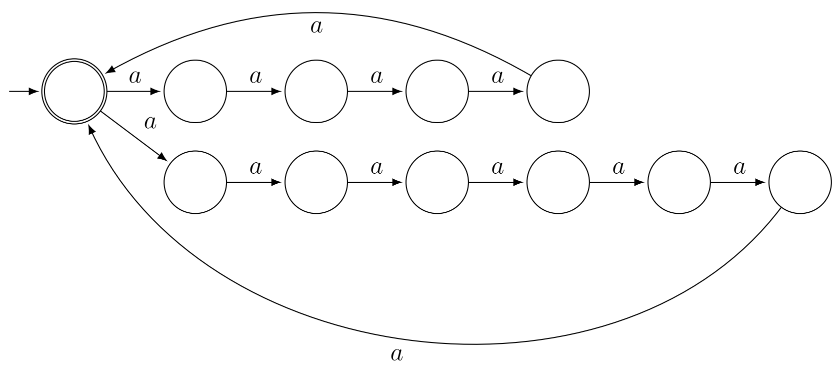
(b) Only (ii) and (iii)

(c) Only (i) and (iii)

(d) Only (ii) and (iv)

(e) All of (i), (ii), (iii), and (iv) ✓

7. Consider the following non-deterministic automata for a unary language $L \subseteq \{a\}^*$ consisting of 11 states with the start state as the unique final state. All transitions shown are on the symbol a.



Let ℓ be the length of the longest string that is NOT accepted by the above automata. Which of the following is TRUE about ℓ?

(a) $\ell = 18$

(b) $\ell = 22$

(c) $\ell = 23$ ✓

(d) $\ell = 26$

(e) ℓ is not well-defined as there are arbitrary long strings that the automata does not accept.

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8. Let $G = (V, E)$ be a weighted, undirected and connected graph, with weight $1 \leq \mathrm{wt}_G(e) \leq 99$ for edge $e \in E$. Suppose G' is the graph with the same set of vertices and edges, but with edge weights $\mathrm{wt}_{G'}(e) = 100 - \mathrm{wt}_G(e)$. Assume that $|V|$ is an even number.

Consider the following statements.

(i) Any minimum-weight spanning-tree of G is also a maximum-weight spanning-tree of G'.

(ii) For any pair of vertices $s, t \in V$, any shortest path from s to t in G is also a longest path from s to t in G'.

(iii) Any minimum-weight perfect matching of G is also a maximum-weight perfect matching of G'.

Which of the above statements ALWAYS TRUE for any such graphs G, G'?

(a) Only (i) and (ii)

(b) Only (i) and (iii) ✓

(c) Only (ii) and (iii)

(d) (i), (ii) and (iii)

(e) None of (i), (ii) or (iii) is always true.

9. Consider the following pseudocode. The program should take as input n, and when it ends, the element $A[i]$ of the vector A should contain the binomial coefficient $\binom{n}{i}$ (with the index of A starting from 0). In order to do so, what should replace the _______ in the pseudocode?

```
procedure BinomialCoefficient
    input n
    for i ← 0 to n do
        A[i] ← 0
    end for
    A[0] ← 1
    for i ← 1 to n do
        B ← A
        for j ← 1 to n do
            A[j] ← _______
        end for
    end for
end procedure
```

(a) $B(j-1) + B(j)$ ✓

(b) $A(j-1) + B(j-1)$

(c) $\dfrac{B(j) + B(n-j)}{2}$

(d) $\dfrac{B(j-1) \times B(j)}{2}$

(e) $\dfrac{A(j-1)+B(j-1)}{2}$

10. Suppose 3 elements are hashed independently and uniformly at random one by one to slots in a hash table of size 6 (assume that in case of a collision, the element is hashed to the first free slot). What is the expected number of elements that DO NOT collide with an existing element in the table?

(a) 25/18

(b) 3/2

(c) 59/36

(d) 2

(e) 5/2 ✓

11. Recall the insertion sort algorithm:

procedure InsertionSort(A)
  for $i \leftarrow 1$ **to** A.length $- 1$ **do**
    $key \leftarrow A[i]$
    $j \leftarrow i - 1$
    while $j \geq 0$ **and** $A[j] > key$ **do**
      $A[j+1] \leftarrow A[j]$
      $j \leftarrow j - 1$
    end while
    $A[j+1] \leftarrow key$
  end for
end procedure

Suppose for some parameter k (possibly a function of the length of the array n), we are guaranteed that the array has k inversions, i.e., there are exactly k pairs (i, j) such that $A[i] > A[j]$ and $i < j$. Then, the running time of insertion sort for A is:

(a) $\Theta(n+k)$ ✓

(b) $\Theta(n \log n + k)$

(c) $\Theta(n^2 + k)$

(d) $\Theta(k \log n)$

(e) $\Theta(n \log k)$

12. Let m, n be positive integers such that $10 \leq m \leq n$. Let A be an $m \times n$ matrix with real entries. Let v be a unit vector in $\mathbb{R}^n$ with the maximum possible value of $\|Av\|_2$. Define $\ell := \|Av\|_2^2$ and let λ be an eigenvalue of A^TA with the largest absolute value. Which of the following is TRUE? (Here, for a vector u, $\|u\|_2$ represents its Euclidean length.)

(a) v may not exist.

(b) v always exists, but λ may not be real.

(c) v **always exists,** λ **is real, and** $\lambda = \ell$**.** ✓

(d) v always exists, λ is real, $\lambda \leq \ell$, and $\lambda \neq \ell$ is possible.

(e) v always exists, λ is real, $\lambda \geq \ell$, and $\lambda \neq \ell$ is possible.

13. There are n people in a house and n pairs of shoes such that the i-th shoe fits the i-th person's feet (and no one else's). A burglar comes and throws the shoes around in some indescribable manner. The people want to determine whose feet fit which shoe and they can only use the following oracles:

- Fit : Takes as input a person and a shoe and outputs $<$ if the person's feet are too small for the shoe, $=$ if they fit, and $>$ if they are too large.
- $\mathsf{Compare}$: Takes as input two people or two shoes and outputs $<$ if the first is smaller than the second, $=$ if they are the same size, and $>$ if the first is larger than the second.

Which of the following statements are TRUE (assuming access to randomness)?

1. If they only have the Fit oracle, the determination can be made in $O(n \log n)$ calls to the oracle in expectation.
2. If they only have the $\mathsf{Compare}$ oracle, the determination can be made in $O(n \log n)$ calls to the oracle in expectation.
3. If they have both oracles, the determination can be made in $O(\sqrt{n})$ calls to the oracles in expectation.

(a) Only 1.

(b) Only 2.

(c) Only 3.

(d) Only 1 and 2. ✓

(e) All three statements are true.

14. Let G be an undirected graph. For any pair of vertices s, t in G, let $\mathsf{MinCut}(s, t)$ be the least number of edges that have to be deleted from G so that there is no s-t path in the resulting graph. Let a, b, c be three vertices in G such that $\mathsf{MinCut}(a, b) \geq \mathsf{MinCut}(b, c) \geq \mathsf{MinCut}(a, c)$. Consider the following statements.

(i) $\mathsf{MinCut}(a, b) > \mathsf{MinCut}(b, c) > \mathsf{MinCut}(a, c)$.

(ii) $\mathsf{MinCut}(a, b) = \mathsf{MinCut}(b, c) > \mathsf{MinCut}(a, c)$.

(iii) $\mathsf{MinCut}(a, b) > \mathsf{MinCut}(b, c) = \mathsf{MinCut}(a, c)$.

(iv) $\mathsf{MinCut}(a, b) = \mathsf{MinCut}(b, c) = \mathsf{MinCut}(a, c)$.

Which of the above statements are possible?

(a) Only (i) and (ii)

(b) Only (ii) and (iii)

(c) Only (iii) and (iv) ✓

(d) Only (ii), (iii), (iv)

(e) All of (i), (ii), (iii), (iv)

15. Let m be a positive integer and a be an integer such that $a \in \{0, 1, \ldots, m-1\}$. Let $S_a = \{b : b \in \{0, 1, \ldots, m-1\}, b^2 \equiv a \mod m\}$ be the set of square roots of a modulo m. Consider the following statements.

(i) For all natural numbers m and for all $a \in \{0, 1, \ldots, m-1\}$, $|S_a| \leq 2$.

(ii) For all natural numbers m, there exists an $a \in \{0, 1, \ldots, m-1\}$, $|S_a| > 2$.

(iii) If m is a prime number, then for all $a \in \{0, 1, \ldots, m-1\}$, $|S_a| \leq 2$.

(iv) For composite m, $|S_a|$ is not necessarily bounded above by 2.

Which of the above statements are TRUE ?

(a) Only (i)

(b) Both (ii), (iv)

(c) Only (iii)

(d) Only (iv)

(e) Both (iii), (iv) ✓

Learning, Information and Data Sciences Section

1. Consider the vector $\vec{v}$ which is closest (in Euclidean distance) to $\begin{pmatrix} a & b & c \end{pmatrix}$ that can be written in the form

$$\vec{v} = \alpha \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

where a, b, c and α, β are real numbers. Recall that the squared (Euclidean) distance between two vectors $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$ and $\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}$ is given by $\sum_{i=1}^{3}(x_i - y_i)^2$. Consider the following statements about the coefficients α and β of the closest vector $\vec{v}$:

(i) $\alpha = (a + 2b + c)/6$

(ii) $\beta = (a + b + c)/3$

(iii) $2\alpha + \beta = b$

Which of the above statements are always TRUE?

(a) (i) only

(b) (ii) only

(c) (iii) only ✓

(d) (i) and (ii) only

(e) None of the above

2. A 10cm long string is cut at two independently and uniformly chosen locations. What is the probability that the resulting three pieces of string can form the sides of a triangle?

(a) 2/3

(b) 1/2

(c) 1/3

(d) 1/4 ✓

(e) None of the above

3. From a survey conducted by a radio station, it was determined that a randomly chosen listener likes Hindustani classical music with probability 0.85, Carnatic music with probability 0.8, film music with probability 0.75, and Western classical music with probability 0.7. What is the *least* possible value for the probability that a randomly chosen listener likes all four forms of music?

(a) 0

(b) 0.1 ✓

(c) 0.225

(d) 0.5

(e) None of the above

4. Suppose in a sequence of tosses of a coin, the first toss is equally likely to show heads or tails. For the next toss, the probability of seeing heads is 2/3 if the first toss was heads and the probability of seeing heads is 1/3 if the first toss was tails. In general, for the $(k+1)$-th toss, the probability of heads is $(1+m)/(2+k)$ where m is the number of heads seen in the first k tosses. What is the expected number of tosses before the first heads is seen?

(a) 2

(b) 2.5

(c) 5

(d) ∞ ✓

(e) None of the above

5. Recall the definitions of the ∞-norm of a vector $x \in \mathbb{R}^n$ and of a matrix $A \in \mathbb{R}^{m\times n}$:

$$\|x\|_\infty = \max_{i=1}^{n} |x_i| \quad \text{and} \quad \|A\|_\infty = \sup_{x:\|x\|_\infty=1} \|Ax\|_\infty .$$

Find the ∞-norm of the following matrix: $\begin{bmatrix} 2 & 3 \\ -2 & 4 \end{bmatrix}$

(a) 4

(b) 7

(c) 6 ✓

(d) 5

(e) 8

6. Suppose X is a 10×10 matrix with complex entries and that $X^3 = X^4$. Which of the following is necessarily TRUE?

(a) $X = X^2$

(b) X is the identity matrix

(c) $X = 0$

(d) X is a real matrix

(e) None of the above ✓

7. Let (X, Y) be an $\mathbb{R}^2$-valued random variable. Suppose $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ and $\mathbb{E}[XY] = \frac{1}{2}$. Which of the following is necessarily TRUE?

(a) $\mathbb{E}[X^4] = 1$

(b) $\mathbb{E}[X^2] = \mathbb{E}Y^2$

(c) $\mathbb{E}[X^2] < e^7$

(d) $\mathbb{E}[X^2] > e^{-7}$

(e) None of the above ✓

8. Let $X = (X_1, X_2)$ be a point chosen uniformly at random from the interior of an origin centered regular hexagon of side length 1. Which of the following is TRUE?

(a) $\mathbb{E}[X_1^2 + X_2^2] \geq 1$

(b) The first and second eigenvalues of the matrix $\mathbb{E}[XX^T]$ are equal ✓

(c) $\mathbb{P}[X_1^2 + X_2^2 > 0.999999] > \frac{1}{10}$

(d) $\mathbb{P}[(X_1^2 + X_2^2)^2 > 0.999999] > \frac{1}{10}$

(e) None of the above

9. Let X, Y, Z be complex $n \times n$ matrices. Define $[X, Y] = XY - YX$. Which of the following is FALSE?

(a) $[X, X] = 0$

(b) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

(c) $[X, Y] = -[Y, X]$

(d) $[[X, X], Y] = 0$

(e) None of the above ✓

10. Let $A, B \subseteq \mathbb{R}^2$ be two-dimensional sets defined as follows.

$$A = \{(x_1, x_2) : \max_{i=1,2} |x_i| \leq 1\},$$

and

$$B = \{(x_1, x_2) : \sum_{i=1}^{2} |x_i| \leq a\},$$

for some $0 < a < 2$. Then what is the maximum number of corners $A \cap B$ can have

(a) 32

(b) 16

(c) 8 ✓

(d) 4

(e) None of the above

11. Consider a six-faced dice that has equal probability of showing any of the six numbers from $\{1, 2, \ldots, 6\}$ on each roll. Let X be the number seen on the first roll and let $Y = 6 - X$. Let $Z = \max\{X, Y\}$. Consider the following statements

(i) $\mathbb{E}[Z] = 4$

(ii) Z is uniformly distributed over $\{3, 4, 5, 6\}$

(iii) $\mathbb{E}[Z] = \frac{9}{2}$

Then which of the following statements is TRUE?

(a) (i) only

(b) (ii) only

(c) (iii) only ✓

(d) Both (i) and (ii)

(e) Both (ii) and (iii)

12. Consider a right-angled triangle ABC, with angle ABC as 90°. Let the height AB be constant, and let the base BC be of length x and hypotenuse AC be of length y. Let x be changing with $dx/dt = 1$, and angle BCA is $\theta(t)$. What is dy/dt?

(a) $\sin(\theta(t))$

(b) $1/\cos(\theta(t))$

(c) $\cos(\theta(t))$ ✓

(d) $1/\sin(\theta(t))$

(e) None of the above

13. For two $n \times n$ matrices A and B, suppose that there exist two non-zero n-dimensional vectors x, y such that $Ax + By = 0$. Which of the following is always TRUE?

(a) A and B are similar (two matrices A and B are similar if there exists a matrix C such that $A = C^{-1}BC$)

(b) A and B have at least one common non-zero eigenvalue

(c) A and B have at least one common eigenvector

(d) A and B have the same rank

(e) None of the above ✓

14. Consider a set S_n in n dimensions defined as

$$S_n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : x_i \geq 0 \ \forall \ i = 1, \ldots, n, \ \sum_{i=1}^{n} x_i = 1\}.$$

Let $diameter(S_n) = \max_{\mathbf{x},\mathbf{y} \in S_n} ||\mathbf{x} - \mathbf{y}||$, where $||\mathbf{x} - \mathbf{y}||$ is the Euclidean distance between $\mathbf{x}, \mathbf{y}$. What is $diameter(S_n)$? [Hint: It might be easier to bound $||\mathbf{x} - \mathbf{y}||^2$.]

(a) $\sqrt{2}$ ✓

(b) 2

(c) $\sqrt{n}$

(d) $\sqrt{3}$

(e) None of the above

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f has at least three distinct real zeros. Define the function $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = f'(x) + xf(x)$. Then g has

(a) no real zeros

(b) exactly one real zero

(c) at least two real zeros ✓

(d) less than two real zeros

(e) none of the above