PART : MATHEMATICS

1. In how many ways, a 5 letter word can be made using any distinct 5 alphabets such that the middle alphabet is 'M' and letter should be in increasing order.

(1) 2198 (2) 4031 (3) 9014 (4) 5148

Ans. **(4)**

Sol. There are 12 alphabets before M and 13 alphabets after M.

So, total number of ways = ${}^{12}C_2 \times {}^{13}C_2 = 66 \times 78 = 5148$

2. The value of $\sum_{r=0}^{5} \frac{{}^{11}C_{2r+1}}{2r+2}$ is :

(1) $\frac{512}{3}$ (2) $\frac{2047}{12}$ (3) $\frac{1023}{12}$ (4) $\frac{2049}{12}$

Ans. **(2)**

Sol. $\frac{1}{12} \sum_{r=0}^{5} \frac{12}{2r+2}\, {}^{11}C_{2r+1} = \frac{1}{12} \sum_{r=0}^{5} {}^{12}C_{2r+2} = \frac{1}{12}\left[{}^{12}C_2 + {}^{12}C_4 + {}^{12}C_6 + {}^{12}C_8 + {}^{12}C_{10} + {}^{12}C_{12}\right]$

$= \frac{1}{12}\left[2^{12-1} - 1\right] = \frac{2047}{12}$

3. If $\sum_{r=0}^{n} T_1 = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{64}$ then find $\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{T_r}$

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) $\frac{1}{6}$ (4) $\frac{1}{2}$

Ans. **(1)**

Sol. $T_n = S_n - S_{n-1}$

$= \frac{(2n-1)(2n+1)(2n+3)(2n+5)-(2n-3)(2n-1)(2n+1)(2n+3)}{64}$

$T_n = \frac{(2n-1)(2n+1)(2n+3)}{8}$

$\frac{1}{T_n} = \frac{8}{(2n-1)(2n+1)(2n+3)}$

$\frac{1}{T_n} = 2\left(\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n-1)(2n+3)}\right)$

T_{r-1} T_r

$S_n = 2\left(\frac{1}{1\times 3} - \frac{1}{(2n-1)(2n+3)}\right)$

$\lim_{n\to\infty} S_n = \frac{2}{3}$

4. If $e^{5(\ell nx)^2+3} = x^8$, then product of all real values of x

(1) $e^{2/5}$ (2) $e^{3/5}$ (3) $e^{8/5}$ (4) $e^{1/5}$

Ans. (3)

Sol. $e^{5(\ell nx)^2+3} = x^8$

$5(\ell nx)^2 + 3 = 8\ell nx$

$\Rightarrow$ $\ell nx = t$

$\Rightarrow$ $5t^2 - 8t + 3 = 0$

$\Rightarrow$ $(t-1)(5t-3) = 0$

$t = \frac{3}{5}, 1$

$\ell nx = \frac{3}{5}$; $\ell nx = 1$

$x = e^{3/5}$; $x = e^1$

Product $= e^{3/5}.e^1 = e^{8/5}$ **Ans.**

5. In a bag there are 6 white and 4 black balls two balls are drawn randomly one by one without replacement then probability that the both balls are white is:

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) $\frac{1}{4}$ (4) $\frac{9}{16}$

Ans. (2)

Sol. Probability $= \frac{{}^6C_2}{{}^{10}C_2} = \frac{6\times5}{10\times9} = \frac{2}{6} = \frac{1}{3}$

6. A be a 3×3 square matrix such that $|A| = -2$ if Det(3adj(–6adj(3A))) $= 2^n \times 3^m$, where $m \geq n$, then $4m + 2n$ is equal to –

Ans. 104

Sol. $|3adj(-6(adj(3A)))|$

$= 3^3\,|adj(-6(adj(3A)))|$

$= 3^3\,|-6\,adj(3A)|^2$

$= 3^3 \times ((-6)^3)^2\,|adj(3A)|^2$

$= 3^9 \times 2^6\,|3A|^4$

$= 3^9 \times 2^6 \times 3^{12}\,|A|^4$

$= 3^{21} \times 2^6 \times 2^4$

$= 3^{21} \times 2^{10}$

$m = 21$ and $n = 10$

So, $4m + 2n = 84 + 20 = 104$.

7. $a_1, a_2, a_3, a_4, \dots$ are positive & increasing terms of G.P. If $a_1 . a_5 = 28$ and $a_2 + a_4 = 29$ then a_6 is equal to

(1) $\sqrt{28}$ (2) $28\sqrt{28}$ (3*) 784 (4) 28

Ans. **(3)**

Sol. Let $a_1 . a_5 = 28$

$a_1 . a_1 r^4 = 28$

$a_1^2 r^4 = 28$ ____(1)

also $a_2 + a_4 = 29$

$a_1 r + a_1 r^3 = 29$

$a_1^2 (r + r^3)^2 = 29^2$

$\frac{28}{r^4}(r + r^3)^2 = 29^2$

$28(1 + r^2)^2 = 841 r^2$

$28 + 28r^4 + 56r^2 = 841r^2$

$28r^4 - 785r^2 + 28 = 0$

$r^2 = \frac{784}{28}, \frac{1}{28}$

$r^2 = 28$ or $\frac{1}{28}$

Now from (1)

$a_1^2 = \frac{28}{28^2} = \frac{1}{28}$

Now $a_6 = a_1 r^5$

$\frac{1}{\sqrt{28}} \times 28^2 \sqrt{28} = 28^2 = 784$

8. Let f(x) be a real differentiable function such that f(0) = 1 and f(x + y) = f(x)f'(y) + f(y) f '(x) for all x, y ∈ R, then $\sum_{n=1}^{100} \log_e f(n)$ is equal to –

Ans. **2525**

Sol. Put x = y = 0

$f(0) = 2f'(0)$ as $f(0) = 1$

$f'(0) = \frac{1}{2}$

Now, put y = 0 in given equation

$f(x) = f(x).f'(0) + f(0).f'(x)$

$f(x) = \frac{1}{2} f(x) + f'(x)$

$f'(x) = \frac{1}{2} f(x)$

$\frac{dy}{dx} = \frac{y}{2}$

$\int \frac{dy}{y} = \int \frac{dx}{2}$

$\ell n y = \frac{x}{2} + C$

$f(0) = 1$

$0 = 0 + C$

$C = 0$

$y = e^{x/2}$

$\ell n y = \frac{x}{2}$

Now, $\sum_{n=1}^{100} \ell n\, f(n) = \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + ... + \frac{100}{2} = \frac{1}{2}\left[\frac{100 \times 101}{2}\right] = 2525$

9. Let the triangle PQR be the image of the triangle with vertices (1, 3) , (3, 1) and (2, 4) in the line $x + 2y = 2$. If the centroid of triangle PQR is the point (α, β) then value of $15\ (\alpha - \beta)$ is:

Ans. **(22)**

Sol. Centroid of the triangle whose vertices are (1, 3) , (3, 1) and (2, 4) is, $\left(2, \frac{8}{3}\right)$.

Image of centroid $\left(2, \frac{8}{3}\right)$ in the line, $x + 2y = 2$ is,

$$\frac{\alpha - 2}{1} = \frac{\beta - \frac{8}{3}}{2} = -2\left(\frac{2 + \frac{16}{3} - 2}{1 + 4}\right)$$

$$\alpha - 2 = \frac{\beta - \frac{8}{3}}{2} = -\frac{2}{5}\left(\frac{16}{3}\right)$$

$$\alpha = -\frac{2}{15}\ ,\ \beta = -\frac{24}{15} \qquad \Rightarrow \qquad \alpha = -\frac{2}{15}\ ,\ \beta = -\frac{24}{15}$$

$$\Rightarrow \qquad 15\ (\alpha - \beta) = 15\left(-\frac{2}{15} + \frac{24}{15}\right) = 22$$

10. (1, 14) and (1, −12) are focii of hyperbola passing through (1, 6), then length of Latus rectum is equal to

(1) $\frac{144}{5}$ (2) $\frac{288}{5}$ (3) $\frac{144}{7}$ (4) $\frac{288}{15}$

Ans. **(2)**

Sol. Let P (1, 6)

By $|PS - PS'| = 2a$

$\left|\sqrt{0 + 64} - \sqrt{0 + 324}\right| = 2a$

$|8 - 18| = 2a$

$a = 5$

and $SS' = 2ae$

$\sqrt{0 + 26^2} = 2ae$

$ae = \frac{26}{2} = 13$

$e = \frac{13}{5}$

Now, $b^2 = a^2\ (e^2 - 1) = 25\left(\frac{169}{25} - 1\right)$

$b = 12$

Length of L.R. $= \frac{2b^2}{a} = \frac{2 \times 144}{5} = \frac{288}{5}$

11. Let z be complex number such that |z| = 1 and z_1, z_2, z_3 are three points satisfying |z| = 1 such that $\arg(z_1) = -\frac{\pi}{4}$, $\arg(z_2) = 0$ and $\arg(z_3) = \frac{\pi}{4}$ also $|z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1| = \alpha + \beta\sqrt{2}$, then $3\beta + 2\alpha =$

(1) 4 (2) 8 (3) 2 (4) 6

Ans. **(1)**

Sol. $z_1 = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1-i}{\sqrt{2}}$

$z_2 = \cos 0 + i \sin 0 = 1$

$z_3 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$

Now,

$|z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1|^2$

$= \left|\frac{1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}} + \left(\frac{1+i}{\sqrt{2}}\right)^2\right|^2$

$= \left|\sqrt{2} - \sqrt{2}i + \frac{1}{2}(2i)\right|^2$

$= \left|\sqrt{2} + i(1 - \sqrt{2})\right|^2$

$= 2 + (1 - \sqrt{2})^2$

$= 2 + 1 + 2 - 2\sqrt{2}$

$= 5 - 2\sqrt{2} = \alpha + \beta\sqrt{2}$

Now $\alpha = 5$ and $\beta = -2$

So $3\beta + 2\alpha$

$= -6 + 10 = 4$

12. Let A = {1,2,3}, then the number of non-empty equivalence relations on set A is :

(1) 4 (2) 6 (3) 8 (4) 5

Ans. **(4)**

Sol. For equivalence relation, relation should be Reflexive, symmetric and transitive:

R_1 = {(1, 1) (2, 2) (3, 3)}

R_2 = {(1, 1) (2, 2) (3, 3) (1, 2) (2, 1)}

R_3 = {(1, 1) (2, 2) (3, 3) (2, 3) (3, 2)}

R_4 = {(1, 1) (2, 2) (3, 3) (1, 3) (3, 1)}

R_5 = {(1, 1) (2, 2) (3, 3) (1, 2) (2, 1), (2, 3), (3, 2), (1, 3) (3, 1)}

13. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $I_1 = \int_0^{\pi/4} f(x)dx$ and $I_2 = \int_0^{\pi/4} xf(x)dx$, then value of $7I_1 + 12I_2$ is:

Ans. (1)

Sol. $f(x) = (7\tan^6 x - 3\tan^2 x).\sec^2 x$

$\therefore \quad I_1 = \int_0^{\frac{\pi}{4}} f(x)\,dx = \int_0^1 \left(7t^6 - 3t^2\right) dt = (t^7 - t^3)_0^1 = 0$

Now $\quad I_2 = \int_0^{\frac{\pi}{4}} xf(x)dx = \int_0^1 \underset{\text{II}}{\left(7t^6 - 3t^2\right)} \underset{\text{I}}{\tan^{-1}t}\, dt$

$= \left(\tan^{-1}t\left(t^7 - t^3\right)\right)_0^1 - \int_0^1 \left(t^7 - t^3\right) \frac{1}{1+t^2} dt$

$= \int_0^1 \frac{t^3\left(1-t^4\right)}{1+t^2}\, dt = \int_0^1 t^3\left(1-t^2\right)\, dt$

$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

Now, $\quad 7I_1 + 12I_2 = 1$

14. Let x(y) is the solution of differential equation $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$ if $x(1) = 1$, then $x(2)$ is equal to:

(1) $\frac{3}{2} + \frac{3}{\sqrt{e}}$ (2) $\frac{3}{2} - \frac{3}{\sqrt{e}}$ (3) $-\frac{3}{2} - \frac{3}{\sqrt{e}}$ (4) $-\frac{3}{2} + \frac{3}{\sqrt{e}}$

Ans. (2)

Sol. $\frac{dx}{dy} = \frac{-x}{y^2} + \frac{1}{y^3}$

$\frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3}$

I.F. $= e^{\int \frac{1}{y^2} dy} = e^{-\frac{1}{y}}$

Now solution of differential equation

$x.e^{-\frac{1}{y}} = \int e^{-\frac{1}{y}}.\frac{1}{y^3} dy + C$

Put $-\frac{1}{y}=t$

$\frac{1}{y^2}dy=dt$

$xe^{-\frac{1}{y}}=-\int e^t.tdt+C$

$xe^{-\frac{1}{y}}=-\left[te^t-e^t\right]+C$

$xe^{-\frac{1}{y}}=e^t(1-t)+C$

$xe^{-\frac{1}{y}}=e^{-\frac{1}{y}}\left(1+\frac{1}{y}\right)+C$

Now when $y=1$ $\Rightarrow$ $x=1$

We get, $1.e^{-1}=e^{-1}(1+1)+C$

$C=-\frac{1}{e}$

Now $xe^{-\frac{1}{y}}=e^{-\frac{1}{y}}\left(1+\frac{1}{y}\right)-\frac{1}{e}$

Put $y=2$

$\frac{x}{\sqrt{e}}=\frac{1}{\sqrt{e}}\left(\frac{3}{2}\right)-\frac{1}{e}$

$x=\frac{3}{2}-\frac{3}{\sqrt{e}}$

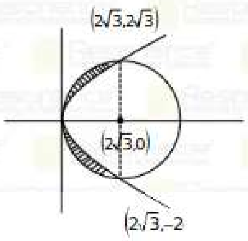
15. The area bounded by inside the circle $\left(x-2\sqrt{3}\right)^2+y^2=12$ and outside the parabola $y^2=2\sqrt{3}\,x$ is

(1) $4(3\pi-8)$ (2) $3(2\pi-5)$ (3) $2(3\pi-8)$ (4) $2(3\pi-5)$

Ans. **(3)**

Sol.

Required area =



$$=2\left[\frac{1}{4}(12\pi)-\int_0^{2\sqrt{3}}\sqrt{2\sqrt{3}x}\right]=2\left[3\pi-\sqrt{2\sqrt{3}}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{2\sqrt{3}}=2\left[3\pi-\sqrt{2\sqrt{3}}\times\frac{2}{3}(2\sqrt{3})^{3/2}\right]$$

$$=2\left[3\pi-\frac{2}{3}\times12\right]=2[3\pi-8]$$

16. Let, $A=\{1,2,3,......,10\}$ and $B=\left\{\frac{m}{n};\, m<n\ \&\ \gcd(m,n)=1\ \&\ m,n\in A\right\}$. Then number of all elements in set B is equal to ________.

Ans. **(31)**

Sol. Number of elements in set B, corresponding to,

m = 1	are		= 9
m = 2	are	n = 3, 5, 7, 9	= 4
m = 3	are	n = 4, 5, 7, 8, 10	= 5
m = 4	are	n = 5, 7, 9	= 3
m = 5	are	n = 6, 7, 8, 9	= 4
m = 6	are	n = 7	= 1
m = 7	are	n = 8, 9, 10	= 3
m = 8	are	n = 9	= 1
m = 9	are	n = 10	= 1

$\therefore$ Total number = 9 + 4 + 5 + 3 + 4 + 1 + 3 + 1 + 1 = 31

17. $f(x)=16(\sec^{-1}x)^2+(\operatorname{cosec}^{-1}x)^2$ then difference between the maximum and the minimum value of, $f(x)$ is equal to ________.

(1) $\frac{1089}{68}\pi^2$ (2) $\frac{1089}{136}\pi^2$ (3) $\frac{1089}{17}\pi^2$ (4) $\frac{1089}{34}\pi^2$

Ans. **(1)**

Sol. $\Rightarrow\ 16(\sec^{-1}x)^2+\left(\frac{\pi}{2}-\sec^{-1}x\right)^2$

$\Rightarrow\ 17(\sec^{-1}x)^2-\pi\sec^{-1}x+\frac{\pi^2}{4}$

$f(x)=17\left(\left(\sec^{-1}x-\frac{\pi}{34}\right)^2\right)+\frac{4\pi^2}{17}$

$f(x)_{max}$ will be at, $\sec^{-1}x=\pi$

i.e. $17\left(\frac{33\pi}{34}\right)^2+\frac{4\pi^2}{17}=\frac{1105}{68}\pi^2$

$f(x)_{min}$ will be at, $\sec^{-1}x=\frac{\pi}{34}$

i.e. $f(x)_{min}=\frac{4\pi^2}{17}$

Now difference of maximum and minimum values of, $f(x)$ is,

$\frac{1105}{68}\pi^2-\frac{4\pi^2}{17}=\frac{1089}{68}\pi^2$

18. Let the parabola $y = x^2 + px - 3$ cuts the coordinate axes at P, Q and R. A circle with centre $(-1, -1)$ passes through P, Q and R, then area of ΔPQR is

(1) 3 (2) 6 (3) 5 (4) 9

Ans. **(2)**

Sol. Parabola cuts x – axis at $y = 0$

$x^2 + px - 3 = 0$

and y –axis at $x = 0$

$y = -3$

Now radius of circle $= \sqrt{(-1-0)^2 + (-1+3)^2} = \sqrt{5}$

equation of circle

$(x+1)^2 + (y+1)^2 = 5$

point of (x, 0) satisfying circle

$(x+1)^2 + 1 = 5$

$x + 1 = \pm 2$

$x = -3, 1$

Now sum of roots of (1) $= -p = -2$

$p = 2$

Solving $x^2 + 2x - 3 = 0$

$x = -3, 1$

So points are $(0, -3)$, $(-3, 0)$ & $(1, 0)$

are $= \frac{1}{2} \times 4 \times 3 = 6$