PART : MATHEMATICS

1. An A.P. consists of 2k elements in which sum of all even terms is 55 and sum of all odd terms is 40 and last term exceeds 1st term by 27 then find k.

Ans. **(5)**

Sol. $a_2 + a_4 + \dots + a_{2k} = 55$ ___(1)

$a_1 + a_3 + a_5 + \dots + a_{2k-1} = 40$ ___(2)

equation (1) – (2)

$d + d + \dots + d = 15$

$kd = 15$

and $t_{2k} = 27 + a$

$a + (2k-1)d = 27 + a$

$2kd - d = 27$

$30 - d = 27 \quad \Rightarrow d = 3$ and $k = 5$

2. 4 boys and 3 girls are to be seated in a row such that all girls sit together and two particular boys B_1 and B_2 are not adjacent to each other. Then the number of ways in which the arrangement can be done.

Ans. **(432)**

Sol. $B_1 \quad B_2 \quad B_3 \quad B_4$

$G_1 \quad G_2 \quad G_3 \quad \rightarrow$ All together

—— $G_1 G_2 G_3$ —— B_3 —— B_4 ——

For B_1 and B_2

${}^4C_2 \times 2! \times 3! \times 3!$

$\frac{4 \times 3}{2} \times 2 \times 6 \times 6 = 432.$

3. If $f(x) = \int_0^{x^2} \frac{t^2 - 8t + 15}{e^t} dt$. Then find number of point of maxima and minima of f(x).

Ans. **(5)**

Sol. $f'(x) = \frac{x^4 - 8x^2 + 15}{e^{x^2}}(2x) - 0$

$f'(x) = \frac{2x(x^2-5)(x^2-3)}{e^{x^2}}$

$f'(x) = \frac{2x(x-\sqrt{3})(x-\sqrt{5})(x+\sqrt{3})(x+\sqrt{5})}{e^{x^2}}$

Number of points = 5.

4. The number of relations, on the set {1, 2, 3} containing (1, 2) and (2, 3), which are reflexive and transitive but not symmetric, is_____

Ans. **(3)**

Sol. R is reflexive $\Rightarrow$ R have (1, 1), (2, 2), (3, 3)

R is transitive

$\because$ (1, 2), (2, 3) $\in R$ $\therefore$ (1, 3) $\in R$

$\therefore R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$

Clearly R_1 is reflexive & transitive but not symmetric

Similarly

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (3, 2)\}$

$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (2, 1)\}$

So four relations.

5. If $2x^2 + (\cos\theta)x - 1 = 0$, $\theta \in [0, 2\pi]$ has roots α and β. Then maximum and minimum value of $\alpha^4 + \beta^4$ is M and m then find 16(M + m)?

Ans. **(25)**

Sol. $2x^2 + (\cos\theta)x - 1 = 0$

$\alpha + \beta = \frac{-\cos\theta}{2}$ $\alpha\beta = -\frac{1}{2}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ $\frac{\cos^2\theta}{4} + 1$

$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = \left(\frac{\cos^2\theta}{4} + 1\right)^2 - \frac{2}{4}$

$\alpha^4 + \beta^4 = \left(\frac{\cos^2\theta}{4} + 1\right)^2 - \frac{1}{2}$

Maximum when $\cos\theta = 1$

$M = \left(\frac{1}{4} + 1\right)^2 - \frac{1}{2}$

$M = \frac{17}{16}$

Minimum when $\cos\theta = 0$

$m = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore$ $16(M+m) = 16\left(\frac{17}{16} + \frac{1}{2}\right) = 25.$

6. Let A and B are two events such that $P(A \cap B) = \frac{1}{10}$ and $P\left(\frac{A}{B}\right)$, $P\left(\frac{B}{A}\right)$ are roots of equation $12x^2 - 7x + 1 = 0$, then $\frac{P(\bar{A} \cup \bar{B})}{P(\bar{A} \cap \bar{B})}$ is equal to:

Ans. $\frac{9}{4}$

Sol. $P\left(\frac{A}{B}\right) . P\left(\frac{B}{A}\right) = \frac{1}{12}$ ...(i)

$\Rightarrow \frac{P(A \cap B)}{P(B)} \frac{P(A \cap B)}{P(A)} = \frac{1}{12}$

$\Rightarrow P(A).P(B) = \frac{12}{100}$ ...(ii)

$P\left(\frac{A}{B}\right) + P\left(\frac{B}{A}\right) = \frac{7}{12}$ ...(iii)

Now, $P\left(\frac{A}{B}\right) + P\left(\frac{B}{A}\right) = \frac{7}{12}$

$\frac{P(A \cap B)}{P(B)} + \frac{P(A \cap B)}{P(A)} = \frac{7}{12}$

$P(A \cap B)\left(\frac{1}{P(B)} + \frac{1}{P(A)}\right) = \frac{7}{12}$ (from ii)

$P(A) + P(B) = \frac{7}{10}$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{7}{10} - \frac{1}{10} = \frac{6}{10}$

Now, $\frac{P(\bar{A} \cup \bar{B})}{P(\bar{A} \cap \bar{B})} = \frac{P(\overline{A \cap B})}{P(\overline{A \cup B})} = \frac{1 - P(A \cap B)}{1 - P(A \cup B)} = \frac{1 - \frac{1}{10}}{1 - \frac{6}{10}} = \frac{9}{4}$

7. If $\theta \in [0, 2\pi]$ satisfy the system of equations $2\sin^2\theta = \cos 2\theta$ and $\cos^2\theta = 3\sin\theta$, then the sum of all real values of θ is -

(1) $\frac{3\pi}{2}$ (2) π (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

Ans. **(2)**

Sol. $2\sin^2\theta = \cos 2\theta$ ...(i) and $\cos^2\theta = 3\sin\theta$ ....(ii)

$2\sin^2\theta = 1 - 2\sin^2\theta$

$4\sin^2\theta = 1$

$\sin\theta = \pm\frac{1}{2}$

$2(1 - \sin^2\theta) = 3\sin\theta$

$2\sin^2\theta + 3\sin\theta - 2 = 0$

$\sin\theta = \frac{1}{2} \Rightarrow \theta \in \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin\theta = -2 \Rightarrow$ Not possible

From above,

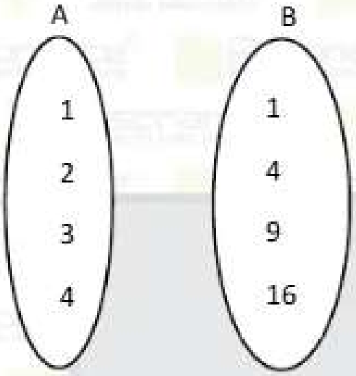
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

8. Let $A = \{1, 2, 3, 4\}$ & $B = \{1, 4, 9, 16\}$; If $f : A \to B$ then number of many one function from A to B are

(1) 127 (2) 151 (3) 232 (4) 280

Ans. **(3)**

Sol.



Total number of functions from $A \to B = 4.4.4.4 = (4)^4$

Total number of one-one function $A \to B = 4!$

Therefore total number of many one function $A \to B$ will be $(4)^4 - 4!$

$= 232$

9. Let $\vec{a}$ and $\vec{b}$ be two unit vector such that angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$. If $\lambda\vec{a} + 3\vec{b}$ and $2\vec{a} + \lambda\vec{b}$ are perpendicular to each other then the product of all possible values of λ is

Ans. **(6)**

Sol. given $|\vec{a}| = 1$

$|\vec{b}| = 1$

$\vec{a} \wedge \vec{b} = \frac{\pi}{3}$

$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

$\frac{1}{2} = \vec{a}.\vec{b}$

$(\lambda\vec{a} + 3\vec{b}).(2\vec{a} + \lambda\vec{b}) = 0$

on solving,

$\lambda^2 + 10\lambda + 6 = 0$

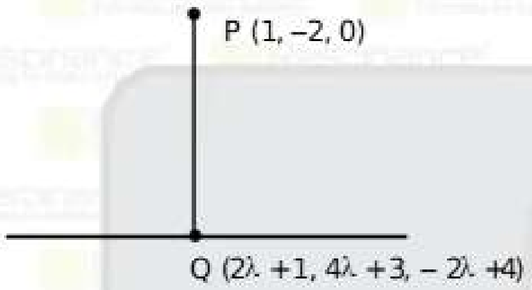
product of roots $= 6$

10. Given a point P (1, –2, 0) and line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-4}{-2}$ find perpendicular distance between the point and line.

(1) $\sqrt{35}$ (2) $\sqrt{\frac{145}{2}}$ (3) $\sqrt{17}$ (4) $\sqrt{\frac{17}{2}}$

Ans. **(1)**

Sol.



$\overrightarrow{PQ}.\vec{c}=0\ \ (\vec{c}=(2,4,-2))$

$2.(2\lambda)+4(4\lambda+5)+(-2)(-2\lambda+4)=0$

$\lambda=-\frac{1}{2}$

Distance between point Q(0, 1, 5) & P (1, –2, 0) = $\sqrt{35}$

11. If system of equations $x+y+2z=6$, $2x+3y+az=a+1$.
$-x-3y+bz=2b$ has infinitely many solution then find value of $7a+3b$.

Ans. **(16)**

Sol. For infinite solution, $\alpha P_1+\beta P_2=P_3$

$\alpha(x+y+2z-6)+\beta(2x+3y+az-a-1)=-x-3y+bz-2b$

$(\alpha+2\beta)x+(\alpha+3\beta)y+(2\alpha+a\beta)z-6\alpha-(a+1)\beta=-x-3y+bz-2b$

Comparing coefficient of x, y, z and constant

$\left.\begin{array}{l}\alpha+2\beta=-1\ \ ...(i)\\ \alpha+3\beta=-3\ \ ...(ii)\end{array}\right]\ \beta=-2 \text{ then } \alpha=3$

$2\alpha+a\beta=b$ ...(iii)

$-6\alpha-(a+1)\beta=-2b$ ...(iv)

Put α, β in equation (iii) and (iv)

$2(3)-2a=b \Rightarrow 6=b+2a$ ...(v)

$-6(3)-(a+1)(-2)=-2b$

$\Rightarrow -18+2a+2=-2b$

$2a+2b=16 \Rightarrow a+b=8$ ...(vi)

By solving (v) and (vi)

$a=-2$ and $b=10$

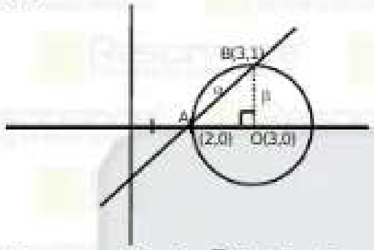
Then $7(-2)+3(10)=16$.

12. If $z(1+i)+\bar{z}(1-i)=4$ divides area bounded by $|z-3|\le 1$ into two areas α and β then find value of $|\alpha-\beta|$.

(1) $\frac{\pi}{2}+1$ (2) $\frac{\pi}{2}-1$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

Ans. **(1)**

Sol.



Line $z(1+i)+\bar{z}(1-i)=4$

$(x+iy)(1+i)+(x-iy)(1-i)=4$

$x-y=2$ ....(i)

circle : $(x-3)^2+y^2\le 1$ ....(ii)

point of intersection x = 2, 3

α = area of sector AB − area of ΔAOB

$=\frac{1}{2}.1^2.\frac{\pi}{2}-\frac{1}{2}.1.1=\frac{\pi}{4}-\frac{1}{2}$

$\beta=\pi(1)^2-\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{3\pi}{4}+\frac{1}{2}$

$|\alpha-\beta|=\left|\frac{\pi}{2}+1\right|=\frac{\pi}{2}+1$

13. If $\sum_{r=1}^{30}\frac{r^2\left({}^{30}C_r\right)^2}{{}^{30}C_{r-1}}=\alpha\cdot 2^{29}$ then find the value of α

Ans. **(465)**

Sol. $\sum_{r=1}^{30}\frac{r^2\left({}^{30}C_r\right)^2}{{}^{30}C_{r-1}}=\sum_{r=1}^{30}r\cdot r\left(\frac{30}{r}\right)\frac{{}^{29}C_{r-1}\cdot{}^{30}C_r}{{}^{30}C_{r-1}}$

$=30\sum_{r=1}^{30}{}^{29}C_{r-1}\cdot r\cdot\left(\frac{{}^{30}C_r}{{}^{30}C_{r-1}}\right)$

$=30\sum_{r=1}^{30}{}^{29}C_{r-1}\cdot((30+1)-r)$

$=30\sum_{r=1}^{30}{}^{29}C_{r-1}\cdot(30-(r-1))$

$=(30)^2\sum_{r=1}^{30}{}^{29}C_{r-1}-(30)\sum_{r=1}^{30}(r-1)\cdot{}^{29}C_{r-1}$

$=(30)^2\cdot 2^{29}-30(29)\cdot 2^{29-1}$

$=\left((30)^2-15\cdot 29\right)2^{29}$

$=(900-435)2^{29}$

$=465\cdot 2^{29}$

$\therefore\ \alpha=465$

14. Find area bounded by curve $y = x^2 - 4x + 4$ and $y^2 = 16 - 8x$.

Ans. **(8/3)**

Sol. Point of Intersection

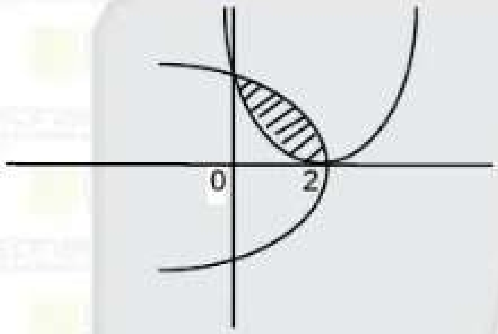
$y^2 = 16 - 8x$

$\left((x-2)^2\right)^2 = 16 - 8x$

$(x-2)^4 = -8(x-2)$

$x = 2$; $x - 2 = -2$

$x = 0$



$$\text{Area} = \int_0^2 \left(\sqrt{16-8x} - (x-2)^2\right)dx = \left(\frac{(16-8x)^{\frac{3}{2}}}{\frac{3}{2}(-8)} - \frac{(x-2)^3}{3}\right)_0^2 = (0-0) - \left(\frac{16^{\cancel{3}/2}}{-12} + \frac{8}{3}\right) = \frac{16}{3} - \frac{8}{3} = \frac{8}{3}$$

15. If tr(M) denotes sum of the diagonal elements of a 3×3 matrix A, $|A| = \frac{1}{2}$ and let $B = \text{adj}(\text{adj}(2A))$ and $\text{tr}(A) = 3$ then find $|B| + \text{tr}(B)$

Ans. **(280)**

Sol. $\text{adj}(\text{adj}A) = |A|^{n-2} \cdot A$

$B = \text{adj}(\text{adj}2A) = |2A|^{(3-2)} \cdot 2A = \left(2^3|A|\right)(2A)$

$= \left(2^3 \times \frac{1}{2}\right)(2A) = 8A$

$|B| = 8^3 . |A| = 256$

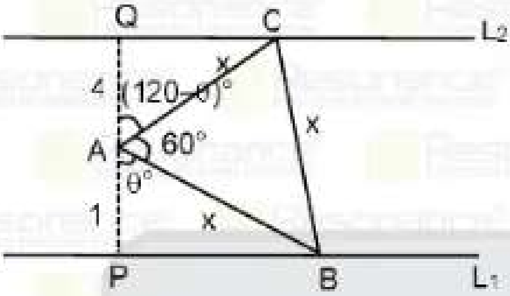
$\text{tr}(B) = 8\,\text{tr}(A) = 8 \times 3 = 24$

$|B| + \text{tr}(B) = 280$

16. Distance between two parallel lines L_1 & L_2 is 5. Find the square of side length of equilateral triangle ABC where B & C are on lines L_1 & L_2 respectively and A is between the lines, one unit away from L_1.

Ans. **(28)**

Sol.



In ΔABP

$$\cos\theta = \frac{1}{x}$$

In ΔACQ

$$\cos(120^\circ - \theta) = \frac{4}{x}$$

$$\left(\frac{-1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right) = \frac{4}{x}$$

$$\left(-\frac{1}{2}.\frac{1}{x} + \frac{\sqrt{3}}{2}\sqrt{1-\frac{1}{x^2}}\right) = 4$$

$\Rightarrow x^2 = 28$

17. $\int e^x\left(\frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{(1-x^2)^{3/2}} + \frac{x}{1-x^2}\right)dx = g(x) + C$. Find $g\left(\frac{1}{2}\right)$

Ans. **($\frac{\pi}{6}\sqrt{\frac{e}{3}}$)**

Sol. Let : $f(x) = \frac{x}{\sqrt{1-x^2}}\sin^{-1}x$

$$f'(x) = \sin^{-1}x\left(\frac{\sqrt{1-x^2} - \frac{x(-2x)}{2\sqrt{1-x^2}}}{(1-x)^2}\right) + \frac{x}{\sqrt{1-x^2}} \times \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) = \sin^{-1}x\left(\frac{1-x^2+x^2}{(1-x^2)^{3/2}}\right) + \frac{x}{1-x^2}$$

$$g(x) = e^x \cdot \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$$

$$g\left(\frac{1}{2}\right) = \frac{\sqrt{e}\times\frac{1}{2}\times\frac{\pi}{6}}{\frac{\sqrt{3}}{2}} = \frac{\pi}{6}\sqrt{\frac{e}{3}}$$