MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. If $2x^2 + (\cos\theta)x - 1 = 0$, $\theta \in [0, 2\pi]$ has roots α and β. Then the sum of maximum and minimum value of $\alpha^4 + \beta^4$.

(1) $\frac{25}{16}$

(2) $\frac{9}{16}$

(3) $\frac{41}{16}$

(4) $\frac{8}{17}$

Answer (1)

Sol. $\alpha + \beta = \frac{-\cos\theta}{2}$

$\alpha\beta = \frac{-1}{2} \Rightarrow \alpha^2\beta^2 = \frac{1}{4}$

$\Rightarrow \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \frac{\cos^2\theta + 1}{4}$

$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$= \left(\frac{\cos^2\theta + 1}{4}\right) - \frac{1}{2}$

For minimum, $\cos\theta = 0$

For maximum, $\cos\theta = 1$

$\Rightarrow$ Minimum $= 1 - \frac{1}{2} = \frac{1}{2}$

Maximum $= \frac{25}{16} - \frac{1}{2} = \frac{17}{16}$

$\Rightarrow$ Sum $= \frac{1}{2} + \frac{17}{16} = \frac{25}{16}$

2. If $\theta \in [0, 2\pi]$ satisfying the system of equations $2\sin^2\theta = \cos 2\theta$ and $2\cos^2\theta = 3\sin\theta$. Then the sum of all real values of θ is

(1) $\frac{3\pi}{2}$ (2) π

(3) $\frac{\pi}{2}$ (4) $\frac{5\pi}{6}$

Answer (2)

Sol. $2\sin^2\theta = \cos 2\theta$

$2\cos^2\theta = 3\sin\theta$

$\Rightarrow$ Adding,

$2 = 1 - 2\sin^2\theta + 3\sin\theta$

$\Rightarrow 2\sin^2\theta - 3\sin\theta + 1 = 0$

$2\sin^2\theta - 2\sin\theta - \sin\theta + 1 = 0$

$2\sin\theta(\sin\theta - 1) - 1(\sin\theta - 1) = 0$

$\sin\theta = 1, \frac{1}{2}$ but $2\sin^2\theta = \cos 2\theta = 2$

but not is not possible

$\Rightarrow \theta = \frac{\pi}{6}, \left(\pi - \frac{\pi}{6}\right)$

$\Rightarrow$ Sum of all values $= \frac{\pi}{6} + \pi - \frac{\pi}{6} = \pi$

3. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16\}$.

If $f : A \to B$, number of many-one functions from A to B are

(1) 24 (2) 232

(3) 256 (4) 252

Answer (2)

Sol. $n(A) = 4$

$n(B) = 4$

Number of many one functions =

Total functions – Number of one-one function

$= 4^4 - 41 = 232$

4. 4 boys and 3 girls are to be seated in a row such that all girls seat together and two particular boys B_1 and B_2 are not adjacent to each other. Then the number of ways in which this arrangement can be done.

(1) 432 (2) 430

(3) 516 (4) 1002

Answer (1)

Sol. 4 boys and 3 girls.

All girls together $\Rightarrow$ $\boxed{G_1\,G_2\,G_3}$

3 girls and 2 boys can be seated in 3!·3! ways

Now B_1 and B_2 go into spaces.

$\Rightarrow {}^4C_2 \times 2! \times 3! \times 3! = 432$

5. If the sum $\sum_{r=0}^{30} \frac{r^2\left({}^{30}C_r\right)^2}{{}^{30}C_{r-1}} = \alpha.2^{29}$, then α is equal to

(1) 225 (2) 465

(3) 345 (4) 425

Answer (2)

Sol. $\frac{r^2 \cdot 30!}{(30-r)!r!} \cdot \frac{30!}{(30-r)!r!} \times \frac{(r-1)!(31-r)!}{30!}$

$= \frac{30!(31-r)}{(r-1)!(30-r)!}$

$= \frac{30(31-r)29!}{(r-1)!(30-r)!}$

$\Rightarrow \sum_{r=0}^{30} \frac{r^2({}^{30}C_r)^2}{{}^{30}C_{r-1}} = 30\sum_{r=0}^{30}(31-r)^{29}C_{r-1} = 30\sum_{r=0}^{30}(31-r)^{29}C_{30-r}$

$= 30\sum_{r=0}^{30}[(30-r)+1]^{29}C_{30-r}$

$= 30\sum_{r=0}^{30}\frac{29}{(30-r)}(30-r).^{28}C_{29-r} + 30\sum_{r=0}^{30}{}^{29}C_{30-r}$

$= 30 \cdot 29 \cdot 2^{28} + 30 \cdot 2^{29}$

$= 30 \cdot 2^{28}(29 + 2) = (31 \times 15) \cdot 2^{29}$

6. Consider a function $f(x) = \int_0^{x^2} \frac{t^2 - 8t + 15}{e^t} dt$. The number of points of extrema are

(1) 3 (2) 5

(3) 7 (4) 9

Answer (2)

Sol. $\because \quad f(x) = \int_0^{x^2} \frac{t^2 - 8t + 15}{e^t}$

$f'(x) = \frac{2x(x^4 - 8x^2 + 15)}{e^{x^2}}$

$= \frac{2x(x^2 - 5)(x^2 - 3)}{e^{x^2}}$

The extremum value of $f(x)$ are $x = 0, \pm\sqrt{5}, \pm\sqrt{3}$

$\therefore$ Number of extremum points are 5.

7. Let A and B are two events such that $P(A \cap B) = \frac{1}{10}$ and $P(A/B)$ and $P(B/A)$ are the roots of the equation $12x^2 - 7x + 1 = 0$, then $\frac{P(\bar{A} \cup \bar{B})}{P(\bar{A} \cap \bar{B})}$ is equal to

(1) $\frac{4}{9}$ (2) $\frac{9}{4}$

(3) $\frac{3}{2}$ (4) $\frac{2}{3}$

Answer (2)

Sol. $P(A \cap B) = \frac{1}{10}$

$P(A/B) + P(B/A) = \dfrac{7}{12}$

and $P(A/B) \cdot P(B/A) = \dfrac{1}{12}$

$$\frac{P(A \cup B)}{P(B)} \cdot \frac{P(A \cap B)}{P(A)} = \frac{1}{12}$$

$$\Rightarrow \; P(A) \cdot P(B) = 12\left(\frac{1}{12}\right)^2 = \frac{12}{100}$$

$$P(A \cap B)\left[\frac{1}{P(A)} + \frac{1}{P(B)}\right] = \frac{7}{12}$$

$$P(A) + P(B) = \frac{7}{10}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{7}{10} - \frac{1}{10} = \frac{6}{10}$$

$$\frac{P(\bar{A} \cup \bar{B})}{P(\bar{A} \cap \bar{B})} = \frac{P(\overline{A \cap B})}{P(\overline{A \cup B})} = \frac{1 - P(A \cap B)}{1 - P(A \cup B)}$$

$$\frac{1 - \frac{1}{10}}{1 - \frac{6}{10}} = \frac{\frac{9}{10}}{\frac{4}{10}} = \frac{9}{4}$$

8. Number of terms in an arithmetic progression is $2n$. Sum of terms occurring at even places is 40 and sum of terms occurring at odd places is 55. If the first term exceeds the last term by 27, then n equals to

(1) 3 (2) 5

(3) 7 (4) 4

Answer (2)

Sol. Let the AP be

$a, a + d, a + 2d, \ldots, a + (2n - 1)d$

Now given that

$(a + d) + (a + 3d) + \ldots + (a + (2n - 1)d) = 40$

$na + n^2d = 40$ …(1)

Also $a + (a + 2d) + (a + 4d) + \ldots + (a + (2n - 2)d) = 55$

$na + dn(n - 1) = 55$ …(2)

Also $a - (a + (2n - 1)d) = 27$

$-(2n - 1)d = 27$ …(3)

$$d = \frac{-27}{2n - 1}$$

(2) – (1)

$dn(n - 1) - n^2d = 15$

$d[n^2 - n - n^2] = 15$

$$\left(\frac{-27}{2n - 1}\right)(-n) = 15$$

$27n = 30n - 15$

$15 = 3n$

$n = 5$

9. The perpendicular distance of point $P(3, 4, 5)$ from the line $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(4\hat{i} - \hat{j} + 5\hat{k})$ is

(1) $\sqrt{\dfrac{19}{42}}$ (2) $\sqrt{\dfrac{19}{21}}$

(3) $\sqrt{\dfrac{42}{19}}$ (4) $\sqrt{\dfrac{21}{19}}$

Answer (1)

Sol. $P(3, 4, 5)$

$L: \vec{r} = \vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(4\hat{i} - \hat{j} + 5\hat{k})$

Any point on L can be. $A(2 + 4\lambda, -1 - \lambda, 1 + 5\lambda)$

Now $\overrightarrow{AP} \cdot (4\hat{i} - \hat{j} + 5\hat{k}) = 0$

$((4\lambda - 1)\hat{i} - (\lambda + 5)\hat{j} + (5\lambda - 4)\hat{k}) \cdot (4\hat{i} - \hat{j} + 5\hat{k}) = 0$

$16\lambda \div 4 + \lambda + 5 + 25\lambda - 20 = 0$

$42\lambda = 19$

$$\lambda = \frac{19}{42}$$

Now $|AP| = \sqrt{(4\lambda - 1)^2 + (\lambda + 5)^2 + (5\lambda + 4)^2}$

$$= \sqrt{\frac{19}{42}}$$

10. In the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, where α, β, γ and δ are the coefficient of x, x^3, x^5 and x^7 respectively. If $\alpha u - \beta v = 18$, $\gamma u + \delta v = 20$ then $u + v$ equal to.

(1) $\frac{-14}{15}$ (2) $\frac{-13}{15}$

(3) $\frac{-3}{5}$ (4) $\frac{-2}{3}$

Answer (1)

Sol. $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$

$$\left[\int_0^3 (x)^5+\int_1^5 x^4\sqrt{x^3-1}+\int_2^5 x^3\left(\sqrt{x^3-1}\right)^2+.....\right]+$$

$$\left[\int_0^5 x^5-\int_1^5 x^4\sqrt{x^3-1}+\int_2^5 x^3\left(\sqrt{x^3-1}\right)^2+.....\right]$$

$$= 2\left[x^5+\int_2^5 x^3(x^3-1)+\int_4^5 x(x^3-1)^2\right]$$

$= 2[x^5 + 10x^6 – 10x^3 + 5x (x^6 + 1 – 2x^3)]$

$= 10x^7 + 20x^6 + 2x^5 – 20x^4 – 20x^3 + 10x$

Coefficient of $x = 10 = \alpha$

Coefficient of $x^3 = –20 = \beta$

Coefficient of $x^5 = 2 = \gamma$

Coefficient of $x^7 = 10 = \delta$

$10u + 20v = 18$ …(i)

$2u + 10v = 20$

$$u=\frac{-11}{3}$$

$$v=\frac{41}{15}$$

$$u+v=\frac{41}{15}-\frac{11}{3}$$

$$=\frac{-14}{15}$$

11. Let $A(6, 8)$, $B(10\cos\alpha, -10\sin\alpha)$ and $C(-10\sin\alpha, -10\cos\alpha)$ be 3 points and if orthocentre of the triangle ABC is (0, 9) then $100\sin^2\alpha$ is equal to

(1) $\frac{25}{4}$ (2) 25

(3) $\frac{15}{4}$ (4) $\frac{5}{4}$

Answer (1)

Sol. Notice, origin is equidistance form *A*, *B* and *C*

$\Rightarrow$ (0, 0) is circumcentre

Since centroid divides orthocentre and circumcentre in 2 : 1 ratio.

$$\Rightarrow \frac{6+10\cos\alpha+(-10\sin\alpha)}{3}=\frac{2(0)+1(0)}{3}=0$$

$\Rightarrow 15\sin\alpha – 10\cos\alpha = 6$

Alos $\frac{8-10\sin\alpha-10\cos\alpha}{3}=\frac{2(0)+1(9)}{3}=3$

$\Rightarrow 8 – 10\sin\alpha – 10\cos\alpha = 9$

$10(\sin\alpha + \cos\alpha) = – 1$

$10(\sin\alpha – \cos\alpha) = – 6$

$20\sin\alpha = 5$

$10\sin\alpha = \frac{5}{2}$

$$100\sin^2\alpha=\frac{25}{4}$$

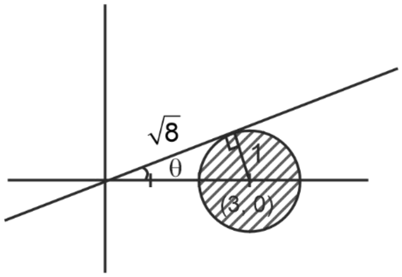
12. If z be a complex number such that $|z – 3| \le 1$, then the equation of line with largest slope passing through origin and z

(1) $x - 2\sqrt{2}y = 0$ (2) $x + 2\sqrt{2}y = 0$

(3) $2\sqrt{2}x + y = 0$ (4) $2\sqrt{2}x - y = 0$

Answer (1)

Sol.



$\tan\theta = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

Therefore, equation of line with maximum slope is

$$(y-0) = \frac{1}{2\sqrt{2}}(x-0)$$

$$\Rightarrow \quad y = \frac{x}{2\sqrt{2}}$$

13. A relation R is defined on set A, $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3)\}$. Elements are added such that R becomes reflexive and transitive but not symmetric. Find the number of such relations.

(1) 3

(2) 4

(3) 2

(4) 9

Answer (1)

Sol. Transitivity

$(1, 2) \in R$, $(2, 3) \Rightarrow (1, 3) \in R$

$(1, 1), (2, 2), (3, 3) \in R$

(2, 1) (3, 2) (3, 1)

(3, 1) cannot be taken.

1. (2, 1) taken and (3, 2) not taken.
2. (3, 2) taken and (2, 1) not taken.
3. Both not taken.

Therefore 3 relations are possible.

14.

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. 21. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$. If $\lambda\vec{a} + 3\vec{b}$ and $2\vec{a} + \lambda\vec{b}$ are perpendicular to each other, then the product of all possible values of λ is ____

Answer (6)

Sol. $|\vec{a}| = 1, |\vec{b}| = 1$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\frac{\pi}{3} = \frac{1}{2}$$

$\lambda\vec{a} + 3\vec{b}$ and $2\vec{a} + \lambda\vec{b}$ are perpendicular

$$\Rightarrow \quad (\lambda\vec{a} + 3\vec{b})\cdot(2\vec{a} + \lambda\vec{b}) = 0$$

$$\Rightarrow \quad 2\lambda + 3\lambda + (\lambda^2 + 6)\vec{a}\cdot\vec{b} = 0$$

$$\Rightarrow \quad 5\lambda + (\lambda^2 + 6)\left(\frac{1}{2}\right) = 0$$

$$\Rightarrow \quad 10\lambda + \lambda^2 + 6 = 0$$

$$\Rightarrow \quad \lambda^2 + 10\lambda + 6 = 0$$

Product of possible values of $\lambda = 6$

22. If A is the 3 × 3 matrix of order 3 × 3, such that $\det(A) = \frac{1}{2}$, $\mathrm{tr}(A) = 10$ and B be another matrix of order 3 × 3 and defined as $B = \mathrm{adj}(\mathrm{adj}(2A))$, then $\det(B) + \mathrm{tr}(B)$ is equal to (where tr(A) denotes trace of matrix A)

Answer (336)

Sol. $B = \mathrm{adj}(\mathrm{adj}2A)$

$B = |2A|^{n-2}(2A)$, [Using $\mathrm{adj}(\mathrm{adj}\,P) = |P|^{n-2}\cdot P$], for $n = 3$

$= |2A|(2A)$

$= 2^3|A|(2A)$

$= 8 \times \frac{1}{2}(2A)$

$= 4(2A)$

$B = 8A$

$|B| = |8A|$

$= 8^3|A|$

$|B| = 8^3 \times \frac{1}{2} = 256$

$B = 8A$

[each element is multiplied 8 times]

$\text{tr}(B) = 8\text{tr}(A)$

$= 80$

$|B| + \text{tr}(B) = 256 + 80$

$= 336$

23. Consider two curves $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with eccentricity e_1 and $E_2 : \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ with eccentricity e_2. If $\frac{e_1}{e_2} = \frac{1}{3}$ and distance between foci of both curves is $2\sqrt{3}$ and $a - A = 4$, then the sum of lengths of latus rectum of both curves is

Answer (12)

Sol. Since, distance between foci

for $E_1 = 2ae$, for $E_2 = 2Ae_2$

$\Rightarrow \; 2ae_1 = 2\sqrt{3} = 2Ae_2$

$\Rightarrow \; ae_1 = Ae_2$

$\Rightarrow \; \frac{e_1}{e_2} = \frac{A}{a} = \frac{1}{3} \Rightarrow a = 3A$

Also $a - A = 4 \Rightarrow A = 2, a = 6$

Now, $2(6)e_1 = 2\sqrt{3} \Rightarrow e_1 = \frac{\sqrt{3}}{6}$

$2(2)e_2 = 2\sqrt{3} \Rightarrow e_2 = \frac{\sqrt{3}}{2}$

$e_1^2 = \frac{1-b^2}{a^2} = \frac{3}{36} = \frac{1-b^2}{36} \Rightarrow \frac{b^2}{36} = \frac{33}{36}$

$b^2 = 33$

$e_2^2 = \frac{1-B^2}{A^2} = \frac{3}{4} = \frac{1-B^2}{4} \Rightarrow B^2 = 1$

Length of latus rectum of $E_1 = \frac{2b^2}{a}$

Length of latus rectum of $E_2 = \frac{2B^2}{A}$

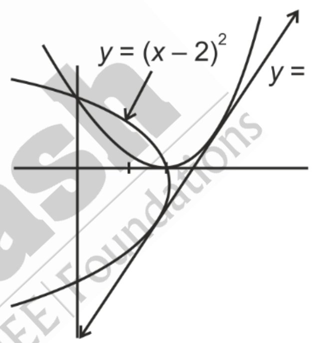
$\Rightarrow$ Sum of lengths of latus rectum

$= \frac{2(33)}{6} + \frac{2(1)}{2} = 11 + 1 = 12$

24. The number of maximum number of common tangents to the curves $y = (x - 2)^2$ and $y^2 = 16 - 8x$ is

Answer (1)

Sol.



Clearly, one tangent is possible. Based on the graphs of these parabola.

25. Let $P(10, -2, -1)$ and Q be the point of perpendicular drawn from point $R(1, 7, 6)$ on the line joining the points $(2, -5, 11)$ and $(-6, 7, -5)$. Then the length PQ is

Answer (13)

Sol. $L_1 : \frac{x-2}{8} = \frac{y+5}{-12} = \frac{z-11}{16}$

$L_1 : \frac{x-2}{2} = \frac{y+5}{-3} = \frac{z-11}{4}$

Avg point of L_1 be $A(2\lambda + 2, -3\lambda - 5, 4\lambda + 11)$

$R(1, 7, 6)$

$$RA.\left(2\hat{i} - 3\hat{j} + 4\hat{k}\right) = 0$$

$$\left((2\lambda + 1)\hat{i} + (-3\lambda - 12)\hat{j} + (4\lambda + 5)\hat{k}\cdot(2\hat{i} - 3\hat{j} + 4\hat{k}\right) = 0$$

$4\lambda + 2 + 9\lambda + 36 + 16\lambda + 20 = 0$

$29\lambda = -58$

$\lambda = -2$

$\therefore$ Foot of perpendicular = $(-2, 1, 3)Q$

$$PQ = \sqrt{(10+2)^2 + (1+2)^2 + (3+1)^2}$$

$= \sqrt{144 + 9 + 16}$

$= 13$

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