PART : MATHEMATICS

1. If the square of the shortest distance between the lines $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+3}{-3}$ and $\frac{x+1}{2}=\frac{y+3}{4}=\frac{z+5}{-5}$ is $\frac{m}{n}$, where m, n are coprime numbers then m + n is equal to:

(1) 6 (2) 9 (3) 14 (4) 21

Ans. **(2)**

Sol. $d=\left|\frac{\overrightarrow{AB}.(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$

$A = (2, 1, -3)$; $B = (-1, -3, -5)$, $\vec{p}=\hat{i}+2\hat{j}-3\hat{k}$, $\vec{q}=2\hat{i}+4\hat{j}-5\hat{k}$

$\overrightarrow{AB}=-3\hat{i}-4\hat{j}-2\hat{k}$

$\overline{p}\times\overline{q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix}=2\hat{i}-\hat{j}$

$d=\left|\frac{-6+4}{\sqrt{4+1}}\right|=\frac{2}{\sqrt{5}}$

$d^2=\frac{4}{5}=\frac{m}{n}$

$m + n = 9$

2. In how many ways 5 boys & 4 girls can sit in a row so that either all boys sit together or no two boys sit together

(1) 5!.4! (2) 4!.4! (3) 3!.4! (4) 6!.4!

Ans. **(4)**

Sol. B_1 B_2 B_3 B_4 B_5

All boys together + no two boys are together

$= (4+1)!\ 5! + 5!\ {}^4C_4 \times 4!$

$= 5!\ 5! + 5!\ 1.\ 4!$

$= 5!\ (5!+4!)$

$= 5!4!\ (5+1) = 6!.4!$

3. Let $f(x) = 6 + 16\cos\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right)\cos x \sin 3x \cos 6x$, if range of f(x) is $[\alpha, \beta]$ then the distance of (α, β) from $3x + 4y + 12 = 0$ is

Ans. **(11)**

Sol. Using these properties

$$4\cos\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right)\cos x = \cos(3x)$$

$2\sin x \cos x = \sin 2x$

$$f(x) = 6 + 16\cos\left(\frac{\pi}{3} - x\right)\cos\left(\frac{\pi}{3} + x\right)\cos x \sin 3x \cos 6x$$

$f(x) = 6 + \sin 12x$

range of $\sin 12x \in [-1, 1]$

range of $f(x) \in [5, 7]$

the distance of point (5, 7) from line $3x + 4y + 12 = 0$

using distance formula

$$\frac{|3(5) + 4(7) + 12|}{\sqrt{3^2 + 4^2}} = 11$$

4. $A = (a_{ij})$ given $A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $\mathbf{A}\begin{bmatrix}4\\1\\3\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $A\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$. The value of a_{23} is________.

(1) 2 (2) –1 (3) 4 (4) –2

Ans. **(2)**

Sol. $A_{3\times3}$

$$\begin{bmatrix}a & b & c\\x & y & z\\\ell & m & n\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow 1 \quad \begin{matrix}b = 0\\y = 0\\m = 1\end{matrix} \quad ......(1)$$

$$\begin{bmatrix}a & b & c\\x & y & z\\\ell & m & n\end{bmatrix}\begin{bmatrix}4\\1\\3\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow 1 \quad \begin{matrix}4a + b + 3c = 0\\4x + y + 3z = \perp\end{matrix} \quad ......(2)$$

$$\begin{bmatrix}a & b & c\\x & y & z\\\ell & m & n\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow 1 \quad \begin{matrix}2a + b + 2c = 1\\2x + y + 2z = 0\\3\ell + m + 2n = 0\end{matrix} \quad ......(3)$$

Eq. (1) and (2) and (3)

$z = -1 = a_{23}$

5. The system of equations $x + y + z = 6$, $x + 2y + 5z = 9$, $x + 5y + \lambda z = \mu$ has no solution if:

(1) $\lambda \neq 17$ and $\mu = 18$ (2) $\lambda \neq 17$ and $\mu \neq 18$

(3) $\lambda = 17$ and $\mu \neq 18$ (4) $\lambda = 17$ and $\mu = 18$

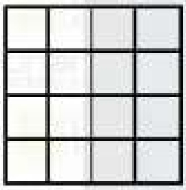
Ans. (3)

Sol.
$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\ 1 & 2 & 5 & 9\\ 1 & 5 & \lambda & \mu\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\ 0 & 1 & 4 & 3\\ 0 & 4 & \lambda-1 & \mu-6\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\ 0 & 1 & 4 & 3\\ 0 & 0 & \lambda-17 & \mu-18\end{array}\right]$$

$\lambda = 17$ and $\mu \neq 18$.

If has no solution λ must be 17 and μ should not be 18.

6. If a square is divided in 4 × 4 squares. If two squares are chosen randomly then the probability that the squares doesn't share common side is –



(1) $\frac{3}{8}$ (2) $\frac{4}{5}$ (3) $\frac{8}{9}$ (4) $\frac{2}{5}$

Ans. (2)

Sol. Total ways for selecting any two squares = ${}^{16}C_2$

Total ways for selecting common side squares

Case I : Horizontal side common = 3 × 4

Case II : Vertical side common = 3 × 4

Probability = $\frac{\text{No. of event occuring}}{\text{Total events}}$

Probability that the squares doesn't share common side = 1 – Probability that the squares share common side

Probability that the squares doesn't share common side = $1 - \frac{24}{{}^{16}C_2}$

Probability = $\frac{4}{5}$

7. If $I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$ then the value of

$\int_0^{2I} \frac{x \sin x \cos x dx}{\sin^4 x + \cos^4 x}$

(1) $\frac{\pi}{16}$ (2) $\frac{\pi^2}{16}$ (3) $\frac{\pi}{8}$ (4) $\frac{\pi^2}{8}$

Ans. (2)

Sol. $I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$

Using property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ (P-5)

$I = \int_0^{\pi/2} \frac{\cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$

Add

$2I = \int_0^{\pi/2} 1dx = \pi/2$

Now, $\int_0^{\pi/2} \frac{x \sin x \cos x dx}{\sin^4 x + \cos^4 x}$

(P-5) & Add.

$\frac{1}{2}\int_0^{\pi/2} \frac{(\pi/2) \sin x \cos x dx}{\sin^4 x + \cos^4 x}$

$\frac{\pi}{4}\int_0^{\pi/2} \frac{\tan x \sec^2 x dx}{\tan^4 x + 1}$

$\tan^2 x = t$

$2\tan x \sec^2 x\, dx = dt$

$\frac{\pi}{8}\int_0^{\infty} \frac{1}{t^2+1} dt = \frac{\pi}{8}\left[\tan^{-1} t\right]_0^{\infty} dt = \frac{\pi}{8} \times \frac{\pi}{2} = \frac{\pi^2}{16}$ **Ans.**

8. $\int x^3 \sin x \, dx = g(x) + c$ then $8g\left(\frac{\pi}{2}\right) + 8g'\left(\frac{\pi}{2}\right) = \alpha\pi^3 + \beta\pi^2 + \gamma$ find $\alpha + \beta - \gamma = ?$

Ans. **(55)**

Sol. $\int \underset{I}{x^3} \underset{II}{\sin x} \, dx = I = g(x) + c$ .....(i)

Applying by parts we get

$I = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$

$\therefore \quad g(x) = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$

$g\left(\frac{\pi}{2}\right) = \frac{3\pi^2}{4} - 6$

Differentiating (i)

$g'(x) = x^3 \sin x$

$g'\left(\frac{\pi}{2}\right) = \frac{\pi^3}{8}$

$8g\left(\frac{\pi}{2}\right) + 8g'\left(\frac{\pi}{2}\right) = \pi^3 + 6\pi - 48$

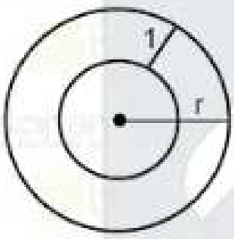
Hence

$\alpha + \beta - \gamma = 55$

9. A chocolate ball is coated with ice of thickness 1 cm. The rate of melting the ice is $\frac{1}{4\pi}$ cm/sec while volume of ice is reducing at a rate of 81 cm^3/sec. The surface area of chocolate is_____.

Ans. **(512π)**

Sol.



$V = \frac{4}{3}\pi r^3$

$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$

$81 = 4\pi \times r^2 \times \frac{1}{4\pi}$

r = 9

Surface area of inner ball is

$4\pi (r - 1)^2 = 4\pi(8)^2 = 512\pi$

10. $y = \left(x - y\frac{dx}{dy}\right)\sin\left(\frac{x}{y}\right)$, x(y) = x and $x(1) = \frac{\pi}{2}$, then the value of cos (x(2)) is:

(1) $2(\log 2)^2 - 1$ (2) $2(\log 2)^2 + 1$ (3) $3(\log 2) - 1$ (4) $3(\log 2) + 1$

Ans. (1)

Sol. $\frac{x}{y} = v$ (let)

$x = vy$

$\frac{dx}{dy} = y\frac{dv}{dy} + v$

$y = \left(x - y\frac{dx}{dy}\right)\sin\left(\frac{x}{y}\right)$

$1 = \left(\frac{x}{y} - \frac{dx}{dy}\right)\sin\left(\frac{x}{y}\right)$

$1 = \left(v - y\frac{dv}{dy} - v\right)\sin v$

$1 = -\left(y\frac{dv}{dy}\right)\sin v$

$-\int\frac{dy}{y} = \int(\sin v)\,dv$

$-\log|y| = -\cos(v) + c$

$\log|y| = \cos\left(\frac{x}{y}\right) - c$

$|y| = e^c \cdot e^{\cos(x/y)}$

$Y = \lambda \cdot e^{\cos(x/y)}$

$1 = \lambda \cdot e^0$ $\quad X(1) = \frac{\pi}{2}$

$Y = e^{\cos(x/y)}$

$\text{Log } y = \cos\left(\frac{x}{y}\right)$

So,

$X(2) = 2\cos^{-1}(\log 2)$

$= \cos(2\cos^{-1}(\log 2))$

$= 2(\cos(\cos^{-1}(\log 2)))^2 - 1$

$= 2(\log 2)^2 - 1$

11. Let $M\left(1,\frac{1}{2}\right)$ be the mid-point of a chord to the ellipse $\frac{x^2}{4}+\frac{y^2}{2}=1$. Then the length of the chord is:

(1) $\frac{2}{3}\sqrt{5}$ (2) $\frac{\sqrt{5}}{3}$ (3) $2\sqrt{\frac{5}{3}}$ (4) $\frac{\sqrt{5}}{2}$

Ans. (3)

Sol. $\because$ Equation of chord is given by $T = S_1$

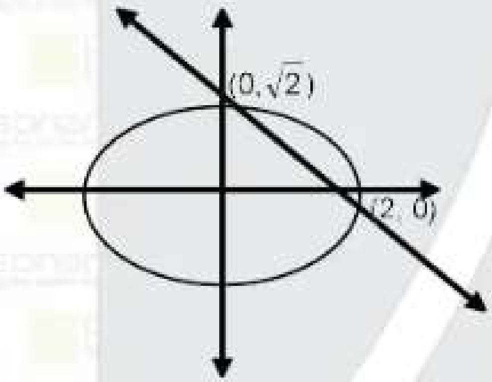
$\Rightarrow$ Equation of chord : $\frac{x.1}{4}+\frac{y.\frac{1}{2}}{2}-1=\frac{1}{4}+\frac{(1/2)^2}{2}-1$

$\Rightarrow \frac{x}{4}+\frac{y}{4}-1=\frac{1}{4}+\frac{1}{8}-1$

$\Rightarrow \frac{x}{4}+\frac{y}{4}=\frac{3}{8}$

$\Rightarrow 2x + 2y = 3.$

$\Rightarrow y = \frac{3-2x}{2}$



On substituting $y = \frac{3-2x}{2}$ in equation of ellipse we will get point of intersection of line and ellipse.

$\Rightarrow \frac{x^2}{4}+\frac{\left(\frac{3-2x}{2}\right)^2}{2}=1 \quad \Rightarrow \quad \frac{x^2}{4}+\frac{9-12x+4x^2}{8}=1$

$\Rightarrow \frac{2x^2+4x^2-12x+9}{8}=1$

$\Rightarrow 6x^2 - 12x + 9 = 8$

$\Rightarrow 6x^2 - 12x + 1 = 0$

$y = \frac{3}{2}-x$

$y_1 - y_2 = \frac{3}{2} - x_1 - \left(\frac{x}{2}-x_2\right) = x_2 - x_1$

$y=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$y=\sqrt{(x_1-x_2)^2+(x_2-x_1)^2}$

$y=\sqrt{2}\sqrt{(x_1+x_2)^2-4x_1x_2}$

$y=\sqrt{2}\sqrt{4-4\times\frac{1}{6}} \quad \Rightarrow \quad y=\sqrt{4-\frac{2}{3}} = \frac{\sqrt{2}\sqrt{10}}{\sqrt{3}} = 2\sqrt{\frac{5}{3}}$

12. α, β are the roots of $x^2 - px + q = 0$ are 10^{th}, 11^{th} terms of A.P. of common difference $\frac{3}{2}$, sum of first 11 terms of this A.P. is 88 then $q - 2p$ is:

Ans. **(158)**

Sol. $\alpha = a + 9d$

$\beta = a + 10d$

$S_{11} = \frac{11}{2}[2a + 10d]$

$2a + 10d = 16$

$a + 5d = 8$

$d = \frac{3}{2}$

$a + 5 \times \frac{3}{2} = 8 \quad \Rightarrow \quad a = \frac{1}{2}$

$\alpha = \frac{1}{2} + 9 \times \frac{3}{2} = \frac{28}{2} = 14$

$\beta = \frac{1}{2} + 10 \times \frac{3}{2} = \frac{31}{2}$

$P = \alpha + \beta$ and $q = \alpha\beta$

$P = 14 + \frac{31}{2}$

$P = \frac{59}{2}$

$q = 14 \times \frac{31}{2}$

$= 217$

Then $q - 2p$

$\Rightarrow$ $217 - 59$

$=$ 158 Ans.

13. Coefficients of x and x^2 in the expansion of $(1 + x)^p (1 - x)^q$ are 1 and –2 respectively. Then $2p - q$ is:

Ans. **(4)**

Sol. $(1+x)^p = (1 + px + \frac{p(p-1)}{2}x^2 + \ldots\ldots\ldots)$

$(1-x)^q = (1 - qx + \frac{q(q-1)}{2}x^2 + \ldots\ldots\ldots)$

$p - q = 1$

$\frac{q(q-1)}{2} + \frac{p(p-1)}{2} - pq = -2$

$q = 2$

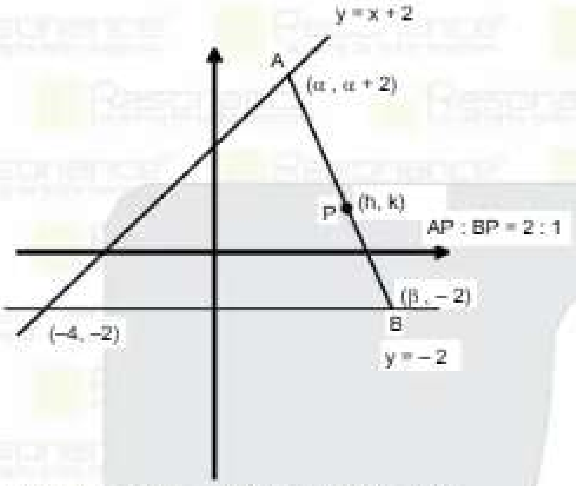
$p = 3$

$q - 2p = -4$

14. A rod AB of length 8 units moving with A on $x - y + 2 = 0$ and B on $y + 2 = 0$. Locus of point P dividing AB in the ratio 2 : 1 internally is $9(x^2 + \alpha y^2 + \beta xy + \gamma x + 28y) - 76 = 0$, then the value of $\alpha - \beta - \gamma$ is:

Ans. **(23)**

Sol.



AB = 8 so AP = 16/3 and BP = 8/3

$(h-\beta)^2 + (k+2)^2 = \frac{64}{9}$ ____(i)

$\frac{2\beta+\alpha}{3} = h$ ____(ii)

$\frac{-4+\alpha+2}{3} = k$ ____(iii)

from (ii) & (iii)

$\beta = \frac{3h-3k-2}{2}$ ____(iv)

from (i) & (iv) eliminate β

$\left(h - \left(\frac{3h-3k-2}{2}\right)\right)^2 + (k+2)^2 = \frac{64}{9}$

$\frac{(3k-h)^2 + 4 + 4(3h-k)}{4} + (k+2)^2 = \frac{64}{9}$

$9(13k^2 + 28k + h^2 - 6hk - 4h + 20) = 4 \times 64$

Replacing (h, k) with (x, y)

$9(13y^2 + 28y + x^2 - 6xy - 4x) - 76 = 0$

$\alpha - \beta - \gamma = 13 + 6 + 4 = 23$

15. $\lim\limits_{x\to\infty}\left(\frac{2x-5}{3x-2}\right)\frac{(3x-1)^{x/2}}{(\sqrt{3x+2})^{x}}$

(1) $\frac{2}{3\sqrt{e}}$ (2) $\frac{3}{2\sqrt{e}}$ (3) $\frac{\sqrt{2}}{3\sqrt{e}}$ (4) $\frac{5}{3\sqrt{e}}$

Ans. **(1)**

Sol. $= \lim\limits_{x\to\infty}\left(\frac{2-\frac{5}{x}}{3-\frac{2}{x}}\right)\left(\frac{3x-1}{3x+2}\right)^{x/2}$

$= \frac{2}{3}\lim\limits_{x\to\infty}\left(\frac{3x-1}{3x+2}\right)^{x/2}$ (1^{∞} form)

$= \frac{2}{3}.e^{\lim\limits_{x\to\infty}\left(\frac{3x-1}{3x+2}-1\right)\frac{x}{2}} = \frac{2}{3}.e^{\lim\limits_{x\to\infty}\left(\frac{3x-1-3x-2}{3x+2}\right)\frac{x}{2}} = \frac{2}{3}.e^{\lim\limits_{x\to\infty}\left(\frac{-3}{2}\right)\left(\frac{1}{3+\frac{2}{x}}\right)} = \frac{2}{3}.e^{-\frac{1}{2}} = \frac{2}{3\sqrt{e}}$

16. Let S be the region consisting of points (x, y) such that $-1 \le x \le 1$ and $0 \le y \le a + e^{|x|} - e^{-|x|}$. If area bounded by the region is $2\left(\frac{e^2+8e+1}{e}\right)$. Then the value of 'a' is:

Ans. **(10)**

Sol. Let $f(x) = a + e^{|x|} - e^{-|x|}$

$f(-x) = a + e^{|x|} - 6^{-|-x|} = a + e^{|x|} - e^{-|x|}$

$\therefore f(x) = f(-x)$

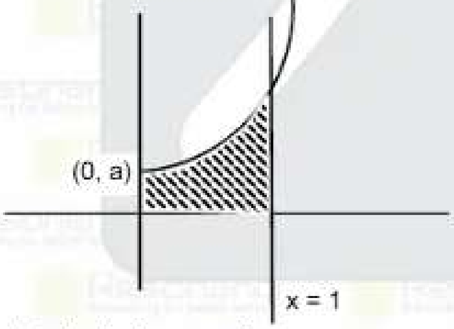
$\therefore$ f(x) is even function

graph of (x) is symmetric about y–axis

$f(0) = a$,

$f'(x) = e^{x} + e^{-x} > 0$ [for $x \ge 0$]

f(x) is increasing [for $x \ge 0$]



Let shaded area = A

required area 2A (graph is symmetric about y-axis)

$a = \int_0^1 (a+e^{x}-e^{-x})dx = \left[ax+e^{x}+e^{-x}\right]_0^1 = \left(a+e+\frac{1}{e}\right)-(2) = A$

required area = 2A = $2\left[\frac{e^2+e(a-2)+1}{e}\right]$

by comparing a – 2 = 8

a = 10

17. Let A and B are two sets such that

$A = \{(x,y) | \quad |x+y| \geq 3,\ x, y \in R\}$;

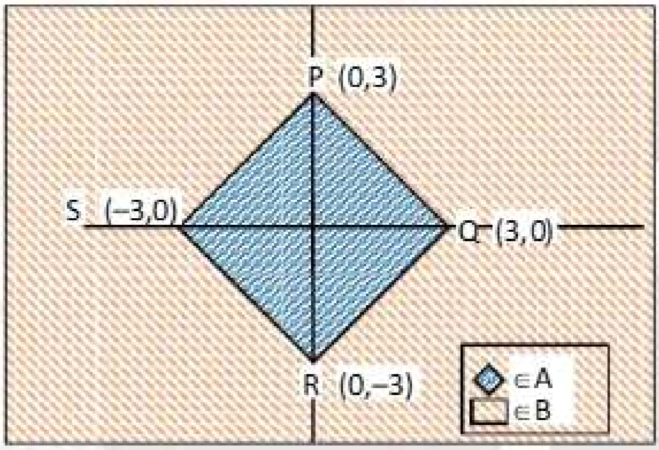
$B = \{(x,y) | \quad |x|+|y| \leq 3,\ x, y \in R\}$

Let $C = A \cap B$

The sum of all possible values of x + y is:

Ans. **(0)**

Sol.



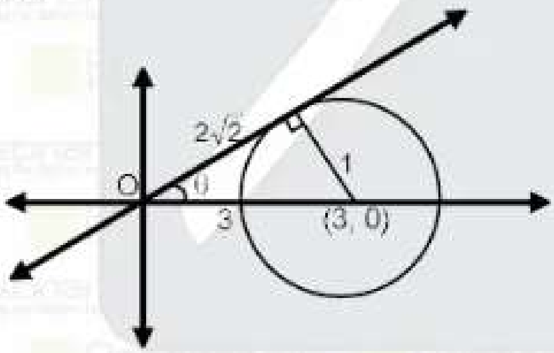
PQ + RS = 0

18. If z = x + iy, x, y ∈ R be a complex number such that $|z - 3| \leq 1$, then the equation of the line with largest slope passing through origin and z :

(1) $x - 2\sqrt{2}y = 0$ (2) $x + 2\sqrt{2}y = 0$ (3) $2\sqrt{2}x + y = 0$ (4) $2\sqrt{2}x - y = 0$

Ans. **(1)**

Sol.



$\tan\theta = \frac{1}{2\sqrt{2}}$

$y = mx$

$\Rightarrow \quad y = \frac{1}{2\sqrt{2}}x$

$\Rightarrow \quad x - 2\sqrt{2}y = 0$