PART : MATHEMATICS

1. If $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{96(x^2+\cos x)}{1+e^x}\,dx = \alpha\pi^3+\beta$, (where α, β are positive integer), then $\alpha+\beta$ is equal to:

(1) 144 (2) 100 (3) 64 (4) 196

Ans. (2)

Sol. $\int_{0}^{\frac{\pi}{2}} \left(\frac{96(x^2+\cos x)}{1+e^x}+\frac{e^x.96(x^2+\cos x)}{e^x+1}\right)dx = \int_{0}^{\frac{\pi}{2}} 96(x^2+\cos x)dx$

$= 96\frac{1}{3}\frac{\pi^3}{8}+96[\sin x]_0^{\pi/2} = 4\pi^3+96$

$\alpha = 4,\quad \beta = 96$

$\alpha+\beta = 100$

2. If $f(x) = \frac{2^x}{2^x+\sqrt{2}}$, $x \in R$, then $\sum_{k=1}^{81} f\left(\frac{k}{82}\right)$ is equal to

(1) $81\sqrt{2}$ (2) 82 (3) $\frac{81}{2}$ (4) 42

Ans. (3)

Sol. $f(x)+f(1-x) = 1$

$\sum_{k=1}^{81} f\left(\frac{k}{82}\right) = f\left(\frac{1}{82}\right)+f\left(\frac{2}{82}\right)+\ldots\ldots+f\left(\frac{40}{82}\right)+f\left(\frac{41}{82}\right)+f\left(\frac{42}{82}\right)+\ldots+f\left(\frac{80}{82}\right)+f\left(\frac{81}{82}\right)$

$= 40+f\left(\frac{1}{2}\right)$

$f\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{\sqrt{2}+\sqrt{2}} = \frac{1}{2} \quad \Rightarrow \quad \sum_{k=1}^{81} f\left(\frac{k}{82}\right) = 40+\frac{1}{2} = \frac{81}{2}$

3. The relation $R = \{(x, y) \mid x, y \in Z, x+y = \text{even}\}$. Then R is:

(1) Equivalence.
(2) Reflexive and Transitive but not Symmetric.
(3) Symmetric and Transitive but not Reflexive.
(4) Reflexive and Symmetric but not Transitive.

Ans. (1)

Sol. $x+y = 2\lambda = \text{even}$

Reflexive: $x+x = 2x = \text{even}$

Symmetric: If $x+y = 2\lambda$, then $y+x$ also 2λ even.

Transitive: $x+y = 2\lambda$

$y+z = 2\mu$

Then, $x+z$

$x+2y+z = 2\lambda+2\mu$

$x+z = 2\lambda+2\mu-2y$

$= 2(\lambda+\mu-y) = \text{even}.$

It is also Transitive.

Hence it is Equivalence.

4. If $\int_0^x t\,f(t)\,dt = x^2 f(x)$ and f(2) = 3, then f(6) is equals to:

(1) 6 (2) 1 (3) 3 (4) 2

Ans. **(2)**

Sol. $\int_0^x t\,f(t)\,dt = x^2 f(x)$

Differentiating both sides,

$x f(x) = x^2 f'(x) + f(x) \cdot 2x$

$-\frac{1}{x} = \frac{f'(x)}{f(x)} (x \neq 0)$

$\log f(x) = -\log x + C$ (integrating both sides)

As $f(2) = 3$,

$\log f(2) = -\log(2) + C$

$\log 3 = -\log(2) + C$

$\Rightarrow C = \log(6)$

Thus, $\log f(x) = -\log x + \log 6$

$\log f(6) = -\log 6 + \log 6 = 0$ (Putting x = 6)

$\log f(6) = \log(1)$

$\Rightarrow f(6) = 1$

5. $z_1 = \sqrt{3} + 2\sqrt{2}i$ and $\sqrt{3}|z_1| = |z_2|$ and $\arg(z_2) = \arg(z_1) + \frac{\pi}{6}$, then area of triangle with vertices z_1, z_2 and origin:

(1) $\frac{11\sqrt{3}}{4}$ (2) $\frac{11\sqrt{3}}{2}$

(3) $\frac{11\sqrt{3}}{5}$ (4) $\frac{11\sqrt{3}}{6}$

Ans. **(1)**

Sol. $\arg(z_2) - \arg(z_1) = 30°$

$\text{Arg}\left(\frac{z_2}{z_1}\right) = 30°$

Given two sides and a angle then area of $\Delta = \frac{1}{2} \times |z_2| \times |z_1| \sin 30°$

$= \frac{1}{2}\sqrt{3} \times |z_1|^2 \times \frac{1}{2}$

$= \frac{1}{2}\sqrt{3} \times 11 \times \frac{1}{2} = \frac{11\sqrt{3}}{4}$

6. $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{33}{65}\right)\right)$ is equal to :

(1) –1 (2) $\frac{\sqrt{3}}{2}$ (3) $-\frac{1}{2}$ (4) 0

Ans. **(4)**

Sol. $\cos\left(\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{33}{56}\right)\right)$

$\cos\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}\right)+\tan^{-1}\left(\frac{33}{56}\right)\right)$

$\cos\left(\tan^{-1}\left(\frac{36+20}{48-15}\right)+\tan^{-1}\left(\frac{33}{56}\right)\right)$

$\cos\left(\tan^{-1}\left(\frac{56}{33}\right)+\tan^{-1}\left(\frac{33}{56}\right)\right)$

$\cos\left(\frac{\pi}{2}\right)=0$

7. If $x^2+|2x-3|-4=0$, then sum of square of roots is:

(1) $6(3-\sqrt{2})$ (2) $3(2-\sqrt{2})$ (3) $6(2-\sqrt{2})$ (4) $3(3-\sqrt{2})$

Ans. **(3)**

Sol. $x^2+2x-3-4=0$; $x\le\frac{3}{2}$

$x^2+2x-7=0$

$x=\frac{-2\pm\sqrt{4+28}}{2}=-1\pm2\sqrt{2}$

$x=2\sqrt{2}-1$ given $x\ge\frac{3}{2}$

$x^2+3-2x-4=0$; $x\le\frac{3}{2}$

$x=\frac{2\pm2\sqrt{2}}{2}$

$\Rightarrow$ $x=1-\sqrt{2}$

then sum of square of roots

$(2\sqrt{2}-1)^2+(1-\sqrt{2})^2$

$=8+1-4\sqrt{2}+1+2-2\sqrt{2}$

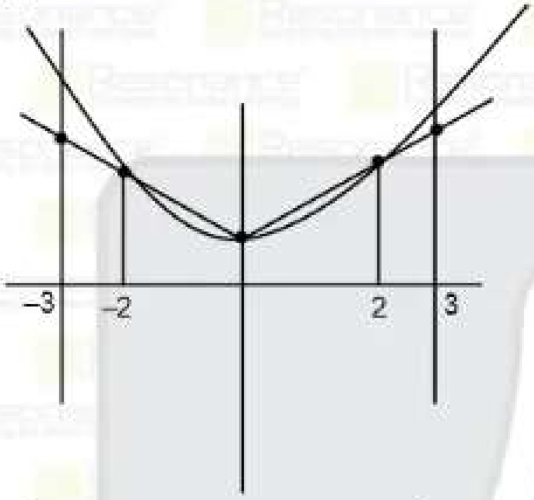
$=12-6\sqrt{2}=6(2-\sqrt{2})$

8. Area of region $\{(x, y) : 0 \le y \le 2|x| + 1,\ 0 \le y \le x^2 + 1,\ |x| \le 3\}$

(1) $\frac{32}{3}$ (2) $\frac{64}{3}$ (3) $\frac{17}{3}$ (4) $\frac{80}{3}$

Ans. **(2)**

Sol.



$x^2 + 1 = 2|x| + 1 \Rightarrow x^2 = 2|x| \Rightarrow x = 0, \pm 2$

$\int_0^2 (x^2+1)dx + \int_2^3 (2x+1)dx$

$\left(\frac{8}{3}+2\right) + [x^2+x]_2^3 = \frac{14}{3} + 12 - 6 = \frac{32}{3}$

By symmetry area will be double.

$\frac{32}{3} \times 2 = \frac{64}{3}$

9. Let $f(x) = (2+3a)x^2 + \left(\frac{a+2}{a-1}\right)x + b,\ a \ne 1$ and

$f(x+y) = f(x) + f(y) + 1 - \frac{2}{7}xy\ \forall\ x, y \in R$, then the value of $28\sum_{i=1}^{5}|f(i)|$ is equal to

Ans. **(675)**

Sol. comparing the value of f(x + y) from both the given equations

$\Rightarrow (2+3a)(x^2+2xy+y^2) + \left(\frac{a+2}{a-1}\right)(x+y) + b = (2+3a)x^2 + \left(\frac{a+2}{a-1}\right)x + b + (2+3a)y^2 + \left(\frac{a+2}{a-1}\right)y + b + 1 - \frac{2}{7}xy$

$\Rightarrow (2+3a)(2xy) = b + 1 - \frac{2}{7}xy \Rightarrow b = -1, \quad a = -\frac{5}{7}$

putting the value of a & b in the first equation

$f(x) = -\frac{x^2}{7} - \frac{3x}{4} - 1 \quad x > 0 \Rightarrow f(x) < 0$

$|f(x)| = \frac{x^2}{7} + \frac{3x}{4} + 1$

$28\sum_{i=1}^{5}|f(i)| = \sum_{i=1}^{5} 4x^2 + 21x + 28 = 4 \times 55 + 21 \times 15 + 28 \times 5 = 675$

10. There are 2 bad oranges mixed with 7 good oranges. Now 2 oranges are drawn at random and let X be the number of bad oranges drawn. The variance of X is

(1) $\frac{1}{18}$ (2) $\frac{65}{162}$ (3) $\frac{49}{162}$ (4) $\frac{16}{81}$

Ans. **(3)**

Sol. Variance in probability $= \sum X^2P(X) - \left(\sum XP(X)\right)^2$

X	0	1	2
P(X)	$\frac{^7C_2}{^9C_2}$	$\frac{^7C_1\,^2C_1}{^9C_2}$	$\frac{^2C_2}{^9C_2}$

$$\text{Variance} = 0^2\left(\frac{^7C_2}{^9C_2}\right) + 1^2\left(\frac{^7C_1\,^2C_1}{^9C_2}\right) + 2^2\left(\frac{^2C_2}{^9C_2}\right) - \left(0\frac{^7C_2}{^9C_2} + 1\frac{^7C_1\,^2C_1}{^9C_2} + 2\frac{^2C_2}{^9C_2}\right)^2$$

$$= 0 + \frac{7}{18} + \frac{1}{9} - \left(\frac{7}{18} + \frac{1}{18}\right)^2$$

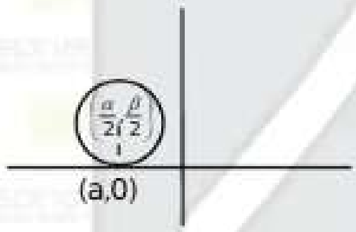
$$= \frac{1}{2} - \frac{16}{81} = \frac{49}{162}$$

11. Let circle $x^2 + y^2 - \alpha x - \beta y + \gamma = 0$ touches X- axis at (a,0) and its Y- intercept is b, then $(2a, b^2)$ in terms of α, β, γ is

(1) $(\alpha, \beta^2 - 4\gamma)$ (2) $(2\alpha, \beta^2 - 4\gamma)$ (3) $(4\alpha, \beta^2 + 4\gamma)$ (4) $(2\alpha, \beta^2 + 4\gamma)$

Ans. **(1)**

Sol.



$a = \frac{\alpha}{2}$

$2a = \alpha$ ..(1)

Y intercepts is b

$2\sqrt{\frac{\beta^2}{4} - \gamma} = b$

$\beta^2 - 4\gamma = b^2$

So $(2a, b^2) = (4\alpha, \beta^2 - 4\gamma)$

12. If ${}^nC_{r-1} = 28$ and ${}^nC_r = 56$, then the value of ${}^nC_{r+1}$ is

Ans. **70**

Sol. **28 =** ${}^8C_2$

${}^nC_{r-1} = {}^8C_2$

On comparing n = 8 and r – 1 = 2 so r = 3

Let's check ${}^nC_r = {}^8C_3 = 56$

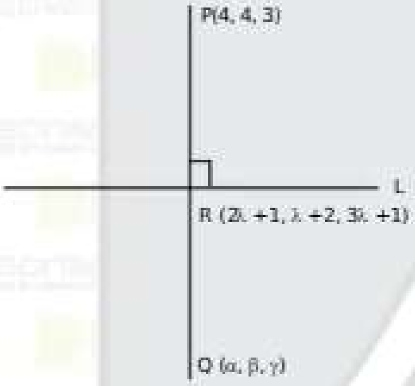
So n = 8 and r = 3 is correct so ${}^nC_{r+1} = {}^8C_4 = 70$

13. If image of point (4, 4, 3) on the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-1}{3}$ is (α, β, γ), then $\alpha + \beta + \gamma$ is

(1) 8 (2) 7 (3) 11 (4) 9

Ans. **(4)**

Sol.



Let $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-1}{3} = \lambda$

Direction ratio of PR : $(2\lambda - 3, \lambda - 2, 3\lambda - 2)$

Direction ratio of L : (2, 1, 3)

PR $\perp$ L

therefore $2(2\lambda - 3) + 1(\lambda - 2) + 3(3\lambda - 2) = 0 \Rightarrow \lambda = 1$

R (3, 3, 4)

Now R is mid point of PQ.

therefore

$\left(\frac{4+\alpha}{2}, \frac{4+\beta}{2}, \frac{3+\gamma}{2}\right) = (3, 3, 4)$

$\alpha = 2, \beta = 2, \gamma = 5$

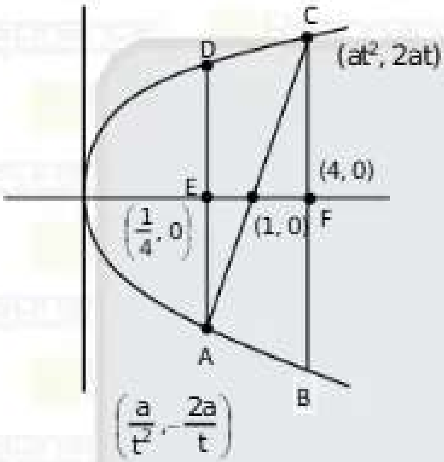
$\alpha + \beta + \gamma = 9$

14. Let ABCD be a trapezium whose vertices lie on parabola $y^2 = 4x$. Let the sides AD and BC of the trapezium be parallel to y-axis. If the diagonal AC is of length $\frac{25}{4}$ and it passes through the point (1, 0) then the area of ABCD is

(1) $\frac{125}{8}$ (2) $\frac{25}{2}$ (3) $\frac{75}{8}$ (4) $\frac{75}{4}$

Ans. (4)

Sol.



Length of focal chord $= a\left(t+\frac{1}{t}\right)^2 = \frac{25}{4}$

$a = 1$

$\left(t+\frac{1}{t}\right) = \pm\frac{5}{2}$

$t = 2, \frac{-1}{2}$

$A = \left(\frac{1}{4}, 1\right)$, $B = (4, -4)$, $C = (4, 4)$, $D = \left(\frac{1}{4}, 1\right)$

$AD = 2$ & $BC = 8$, $EF = \frac{15}{4}$

Area of trapezium

$= \frac{1}{2}(\text{sum of parallel sides}) \times \text{height}$

$= \frac{1}{2}(AD + BC) \times EF = \frac{1}{2}(10)\left(\frac{15}{4}\right) = \frac{75}{4}$

15. Number of ways to form 5 digit numbers greater than 50000 with the use of digits 1,2,3,4,5,6,7,8 such that sum of first and last digit is not more than 8 is equal to

(1) 3072 (2) 5120 (3) 4067 (4) 4068

Ans. **(1)**

Sol. Case-1: 5 is at the ten thousandth place

| 5 | | | | | [1, 2, 3]

Total number of ways $= 8^3 \times 3 = 1536$

Case-2: 6 is at the ten thousand place

| 6 | | | | | [1, 2]

Total number of ways $= 8 \times 8 \times 8 \times 2 = 1024$

Case-3: 7 is at the ten thousand place

| 7 | | | | | [1]

Total number of ways $= 8^3 \times 1 = 512$

$= 3072$

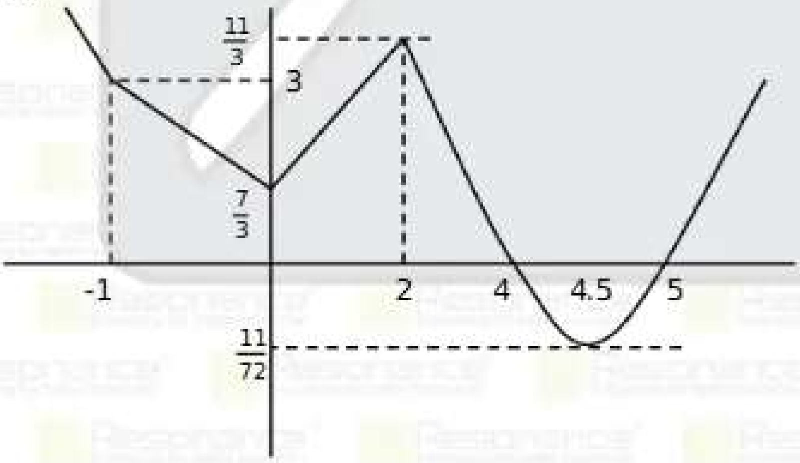
16. The sum of all local minimum values of the function

$$\begin{cases} 1-2x\ ; & x<-1 \\ \frac{1}{3}(7+2|x|)\ ; & -1\le x\le 2, \\ \frac{11}{18}(x-4)(x-5)\ ; & x>2 \end{cases}$$ is equal to

(1) $\frac{421}{72}$ (2) $\frac{157}{72}$ (3) $\frac{9}{2}$ (4) $\frac{13}{2}$

Ans. **(2)**

Sol.



Local minima $= \frac{7}{3} - \frac{11}{72} = \frac{157}{72}$

17. Let T_r be the r^{th} term of an A.P. If for some m, $T_m = \frac{1}{25}$, $T_{25} = \frac{1}{20}$, and $20\sum_{r=1}^{25} T_r = 13$, then the value of $5m\sum_{r=m}^{2m} T_r$ is equal to:

(1) 90 (2) 126 (3) 142 (4) 112

Ans. **(2)**

Sol. $T_r = a + (r-1)d$

$T_m = a + (m-1)d = \frac{1}{25}$

$T_{25} = \frac{1}{20}$

$a + 24d = \frac{1}{20}$

$20\sum_{r=1}^{25} T_r = 13$

$20 \times \frac{25}{2}[2a + 24d] = 13$

$250a + \frac{25}{2} = 13$

$a = \frac{1}{500}$

$\frac{1}{500} + 24d = \frac{1}{20}$

$24d = \frac{1}{20} - \frac{1}{500}$

$d = \frac{1}{500}$

So, a = d

$T_m = a + (m-1)\,d = \frac{1}{25}$

$a\,[1 + m - 1] = \frac{1}{25}$

$m = \frac{1}{25a} = 20$

$5m\sum_{r=m}^{2m} T_r = 5m\left[\frac{(m+1)}{2}(2T_m + md)\right]$

$= 5m\frac{(m+1)}{2}\left[\frac{2}{25} + md\right]$

$= 5m\frac{(m+1)}{2}\left[\frac{2}{25} + \frac{1}{25a}a\right]$

$= 5m\frac{(m+1)}{2}\left[\frac{3}{25}\right] = 126.$

18. Let A(x, y, z) be a point in XY-plane, which is equidistant from three points (0, 3, 2), (2, 0, 3) and (0, 0, 1). If B(1, 4, –1) and C(2, 0, –2), then among the following statements

S_1 : ΔABC is an isosceles right angle triangle.

S_2 : The area of ΔABC is $\frac{9\sqrt{2}}{2}$.

(1) Only S_1 is true (2) Both are false (3) Only S_2 is true (4) Both are true

Ans. **(1)**

Sol. Let point A(x, y, 0),

Let, D (0, 3, 2), E(2, 0, 3) and F (0, 0, 1).

AD = AE = AF

$x^2 + (y-3)^2 + 4 = (x-2)^2 + y^2 + 9 = x^2 + y^2 + 1$

AD = AF $\quad x^2 + (y-3)^2 + 4 = x^2 + y^2 + 1 \quad \Rightarrow y = 2$

AE = AF $\quad (x-2)^2 + y^2 + 9 = x^2 + y^2 + 1$

$\Rightarrow -4x + 13 = 1$

$x = 3$

A (3, 2, 0)

AB = 3

BC = $3\sqrt{2}$

AC = 3

$\because AB^2 + AC^2 = BC^2$ and AB = AC

therefore ABC is right angled isosceles triangle

Now area of triangle ABC is $\frac{1}{2} \times 3 \times 3 = \frac{9}{2}$