PART : MATHEMATICS

1. How many six letter words can be formed by using the letters of the word MATHS, such that each letter used atleast two times.

Ans. **(1405)**

Sol. $^5C_3 \times \frac{6!}{2!2!2!} + {}^5C_2\left(\frac{6!}{2!4!} \times 2 + \frac{6!}{3!3!}\right) + {}^5C_1 . 1$

$= 10 \times 90 + 10\ (15 \times 2 + 20) + 5$

$= 900 + 500 + 5 = 1405$

2. $80\int_0^{\frac{\pi}{4}} \frac{\sin\theta + \cos\theta}{9 + 16\sin 2\theta} d\theta =$

(1) $4\ell n3$ (2) $2\ell n3$ (3) $\ell n3$ (4) $3\ell n9$

Ans. **(1)**

Sol. $\sin\theta - \cos\theta = t$

$(\cos\theta + \sin\theta)\, d\theta = dt$

$1 - \sin 2\theta = t^2$

$80\int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)}$

$80\int_{-1}^{0} \frac{dt}{25 - 16t^2}$

$\frac{80}{10} \times \frac{1}{4}\ell n\left(\frac{5 + 4t}{5 - 4t}\right)\Bigg]_{-1}^{0}$

$2\left[\ell n1 - \ell n\left(\frac{1}{9}\right)\right]$

$= 4\ell n3$

3. Let $f(x) = \begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4\sin 4x \\ \sin^2 x & 1 + \cos^2 x & 4\sin 4x \\ \sin^2 x & \cos^2 x & 1 + 4\sin 4x \end{vmatrix}$

If maximum and minimum values of f(x) are m & n respectively then the value of $m^4 - n^4$ is.

(1) 8704 (2) 1312 (3) 10016 (4) 1280

Ans. **(4)**

Sol. $C_1 \to C_1 + C_2$

$f(x) = \begin{vmatrix} 2 & \cos^2 x & 4\sin 4x \\ 2 & 1 + \cos^2 x & 4\sin 4x \\ 1 & \cos^2 x & 1 + 4\sin 4x \end{vmatrix}$

$R_2 \rightarrow R_2 - R_1$

$$f(x) = \begin{vmatrix} 2 & \cos^2 x & 4\sin 4x \\ 0 & 1 & 0 \\ 1 & \cos^2 x & 1+4\sin 4x \end{vmatrix}$$

$f(x) = 2(1+4\sin 4x) - 4\sin 4x$

$f(x) = 2+4\sin 4x$

$m = 6, \quad n = -2$

$m^4 - n^4 = 6^4 - (-2)^4 = 1296 - 16 = 1280$

4. Let R be a relation as defined by $xRy \Rightarrow \sec^2 x - \tan^2 y = 1$, where $x, y \in [0, \pi/2)$, then the relation R is –

(1) Reflexive but not Symmetric. (2) Symmetric but not Transitive.

(3) Transitive but not Reflexive. (4) Equivalence.

Ans. **(4)**

Sol. Ref. $xRx \Rightarrow \sec^2 x - \tan^2 x = 1$

Symm. $xRy \Rightarrow \sec^2 x - \tan^2 y = 1$

$\Rightarrow 1 + \tan^2 x - (\sec^2 y - 1) = 1$

$\Rightarrow -(\sec^2 y - \tan^2 x) = -1$

$\Rightarrow \sec^2 y - \tan^2 x = 1$

$\Rightarrow yRx$

Transitive

$xRy \Rightarrow \sec^2 x - \tan^2 y = 1$

$yRz \Rightarrow \sec^2 y - \tan^2 z = 1$

Add

$\sec^2 x + 1 - \tan^2 z = 2$

$\sec^2 x - \tan^2 z = 1$

The relation R is symmetric, reflexive and transitive, so the relation R is equivalence.

5. Sum of first three terms of an AP with integral common difference is 54 and sum of first twenty terms lies between 1600 to 1800, then the 11^{th} term of this AP is

(1) 108 (2) 90 (3) 111 (4) 115

Ans. **(2)**

Sol. Let the terms of AP be a-d, a, a + d, ...

$a-d + a + a + d = 54$

$\Rightarrow a = 18$

$T_{20} = (a-d) + 19d$

$1600 < S_{20} < 1800$

$1600 < \frac{20}{2}$ [1^{st} term + last term] < 1800

$1600 < 10\,[(a - d) + a + 18d] < 1800$

$160 < 2a + 17d < 180$

$160 - 2a < 17d < 180 - 2a$

$160 - 36 < 17d < 180 - 36$

$124 < 17d < 144$

$7.29 < d < 8.47$

$\therefore d = 8$ [$\because$ d is an integer]

$T_{11} = (a-d) + 10d$

$T_{11} = a + 9d$

$T_{11} = 18 + (9 \times 8) = 90$

6. $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} - 5\hat{j} + \hat{k}$. If $\vec{a}\times\vec{c} = \vec{c}\times\vec{b}$ and $(\vec{a}+\vec{c}).(\vec{b}+\vec{c}) = 168$ then maximum value of $|\vec{c}|^2$ is

Ans. **(308)**

Sol. $\vec{a}\times\vec{c} = -\vec{b}\times\vec{c}$

$(\vec{a}+\vec{b})\times\vec{c} = \vec{0}$

$\vec{c} = \lambda(\vec{a}+\vec{b})$

$\vec{c} = \lambda(5\hat{i} - 6\hat{j} + 4\hat{k})$

$|\vec{c}|^2 = \lambda^2.77$

$\vec{a}.\vec{b} + \vec{a}.\vec{c} + \vec{b}.\vec{c} + |\vec{c}|^2 = 168$

$14 + \vec{c}.(\vec{a}+\vec{b}) + |\vec{c}|^2 = 168$

$14 + \lambda|\vec{a}+\vec{b}|^2 + 77\lambda^2 = 168$

$14 + 77\lambda + 77\lambda^2 = 168$

$77\lambda + 77\lambda^2 = 154$

$\lambda^2 + \lambda - 2 = 0$

$\lambda = -2, 1$

So maximum value of $|\vec{c}|^2 = (-2)^2 \times 77 = 4 \times 77 = 308$

7. $S = \{x : \cos^{-1}x = \pi + \sin^{-1}x + \sin^{-1}(2x-1)\}$, then $\sum_{x\in S}(2x-1)$ is equal to:

Ans. **(0)**

Sol. $\cos^{-1}x = \pi + \sin^{-1}x + \sin^{-1}(2x-1)$

$\frac{\pi}{2} - \sin^{-1}x = \pi + \sin^{-1}x + \sin^{-1}(2x-1)$

$-\frac{\pi}{2} - 2\sin^{-1}x = \sin^{-1}(2x-1)$

$\sin\left(-\frac{\pi}{2} - 2\sin^{-1}x\right) = \sin(\sin^{-1}(2x-1))$

$-\cos(2\sin^{-1}x) = (2x-1)$

$-(1-2x^2) = 2x-1$

$2x^2 - 2x = 0$

$x = 0, 1$

So, $\sum_{x\in S}(2x-1) = -1 + 1 = 0$

8. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3 + 6k^2 + 11k + 5}{(k+3)!}$ is equal to –

(1) $\frac{4}{3}$ (2) $\frac{5}{3}$ (3) $\frac{7}{3}$ (4) 5

Ans. **(2)**

Sol. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k^3+6k^2+11k+6-1}{(k+3)!}$

$\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{(k+1)(k+2)(k+3)-1}{(k+3)!}$

$\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left[\frac{k!(k+1)(k+2)(k+3)-k!}{k!(k+3)!}\right]$

$\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left[\frac{(k+3)!-k!}{k!(k+3)!}\right]$

$\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left[\frac{1}{k!}-\frac{1}{(k+3)!}\right]$

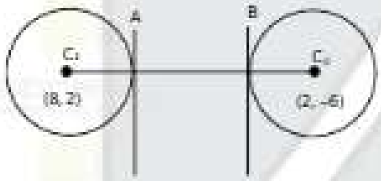
$\lim\limits_{n\to\infty} \left[\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}-\frac{1}{(n+3)!}-\frac{1}{(n+2)!}-\frac{1}{(n+1)!}\right]$

$1+\frac{1}{2}+\frac{1}{6}=\frac{5}{3}$

9. Let z_1 and z_2 are two complex numbers such that $|z_1-8-2i|\le 1$ and $|z_2-2+6i|\le 2$ then minimum value of $|z_1-z_2|$ is equal to

Ans. **(7)**

Sol.



$r_A=1$

$r_B=2$

$C_1C_2=\sqrt{(8-2)^2+(2-(-6))^2}=10$

minimum value of $|z_1-z_2|=C_1C_2-(r_A+r_B)$

$=10-3=7$

10. Number of 7 digit numbers made with digit 1, 2, 3 such that sum of the digits is 11 is equal to –

Ans. **(161)**

Sol. **Case – I** 3 2 2 1 1 1 1 $\Rightarrow \frac{7!}{2!4!}=\frac{7\times6\times5}{2}=105$

Case – II 2 2 2 2 1 1 1 $\Rightarrow \frac{7!}{4!3!}=\frac{7\times6\times5}{6}=35$

Case – III 3 3 1 1 1 1 1 $\Rightarrow \frac{7!}{5!2!}=\frac{7\times6}{2}=21$

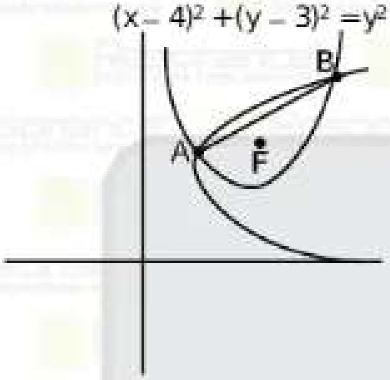
Total number of 7 digit number $=105+36+21=161$

11. Two parabola has foci (4, 3) and directrix of one parabola is X-axis and other parabola is Y-axis. They are intersecting at points A and B. Then the value of $(AB)^2$ is:

Ans. **(192)**

Sol. Equation of parabolas are:

$$(x-4)^2 + (y-3)^2 = x^2$$
$$(x-4)^2 + (y-3)^2 = y^2$$



Subtracting equation (ii) from equation (i)

$x^2 - y^2 = 0$

$\Rightarrow \quad y = -x,\ y = x$

By $\quad y = -x$, equation has Imaginary roots

By taking $\quad y = x$

$$x^2 - 14x + 25 = 0 \quad \begin{matrix} x_1 \\ x_2 \end{matrix}$$

$$y^2 - 14y + 25 = 0 \quad \begin{matrix} y_1 \\ y_2 \end{matrix}$$

$(AB)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 + (y_1 + y_2)^2 - 4y_1y_2 = (196 - 100) \times 2 = 192.$

12. Find least natural value of n such that number of integral terms in expansion of $\left(\sqrt[3]{7} + \sqrt[12]{11}\right)^n$ are 183.

(1) 2160 (2) 2172 (3) 2184 (4) 2196

Ans. **(3)**

Sol. $T_{r+1} = {}^nC_r\, 7^{\left(\frac{n-r}{3}\right)} 11^{\left(\frac{r}{12}\right)}$

$r = 12k$ and $n = 3m$

$r = 0, 12, 24, 36, \ldots\ldots\ldots\ldots$

$T_{183} = 0 + 182 \times 12 = 2184$

$n = 2184$

13. Area enclosed by $y \geq |x-1|$, $y + |x| \leq 3$, $x^2 \leq 2y - 3$ is A. Then 6A is (in sq. units) –

(1) 5 (2) 10 (3) 20 (4) 15

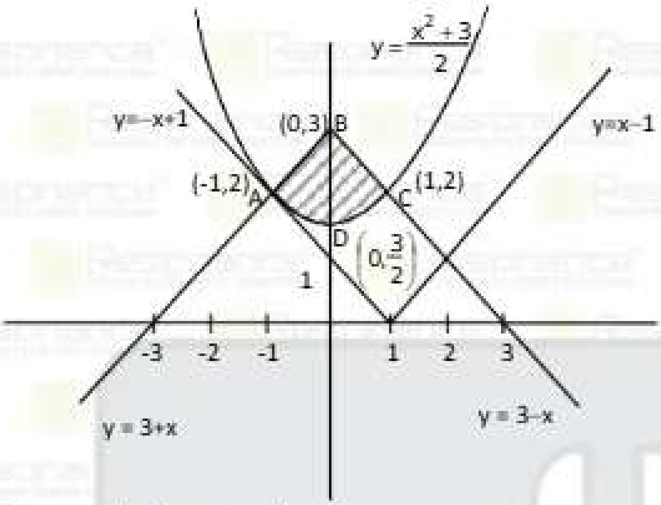
Ans. **(2)**

Sol. $y \geq |x - 1|$

$y \leq 3 - |x|$

$y \geq \dfrac{x^2 + 3}{2}$

As per the question, shaded area is the required area.



Symmetric about y axis.

$$\text{Required area} = 2\left[\int_{3/2}^{2}\sqrt{2y-3}\,dy + \int_{2}^{3}(3-y)\,dy\right]$$

$$2\left[\frac{2.(2y-3)^{3/2}}{2.3}\right]_{3/2}^{2} + 2\left[3y-\frac{y^2}{2}\right]_{2}^{3}$$

$$\left[\frac{2}{3}(1-0)\right] + 2\left[\left(9-\frac{9}{2}\right)-\left(6-\frac{4}{2}\right)\right]$$

$$\frac{2}{3}+2.\frac{9}{2}-8=\frac{2}{3}+1=\frac{5}{3}=A$$

So $\quad 6A = 6\times\frac{5}{3} = 10$

14. If $(\cos x)(\ell n \cos x)^2\,dy + (\sin x - 3y\sin x\,\ell n\cos x)\,dx = 0$ and $y\left(\frac{\pi}{4}\right) = \frac{-1}{\ell n 2}$, then $y\left(\frac{\pi}{6}\right)$ is equal to

(1) $\frac{1}{\ell n 3 - 2\ell n 2}$ (2) $\frac{1}{2\ell n 2 - \ell n 3}$ (3) $\frac{1}{2\ell n 2}$ (4) $\frac{1}{\ell n 3}$

Ans. **(1)**

Sol. $\frac{dy}{dx} - \frac{3\tan x}{\ell n\cos x}.y = \frac{-\tan x}{(\ell n\cos x)^2}$

$\text{I.F.} = e^{\int\frac{-3\tan x}{\ell n\cos x}dx}$ $\qquad \ell n\cos x = t \quad \Rightarrow \quad -\tan x\,dx = dt$

$= e^{3\int\frac{dt}{t}} = e^{3\ell n t} = t^3.$

$\text{I.F} = (\ell n\cos x)^3$

$y.(\ell n\cos x)^3 = \int\frac{-\tan x}{(\ell n\cos x)^2}.(\ell n\cos x)^3 dx$

$= \int -(\tan x)(\ell n\cos x)dx$

$\ell n\cos x = t \quad \Rightarrow \quad -\tan x\,dx = dt$

$= \int t\,dt = \frac{t^2}{2} + C$

$y.(\ell n\cos x)^3 = \dfrac{(\ell n\cos x)^2}{2} + C$

$$y\left(\frac{\pi}{4}\right) = -\frac{1}{\ell n2} \Rightarrow \left(\ell n\frac{1}{\sqrt{2}}\right)^3\left(-\frac{1}{\ell n2}\right) = \frac{\left(\ell n\frac{1}{\sqrt{2}}\right)^2}{2} + C$$

$$\frac{1}{8}(\ell n2)^2 = \frac{1}{8}(\ell n2)^2 + C$$

$C = 0$

$$y = \frac{1}{2\ell n\cos x}$$

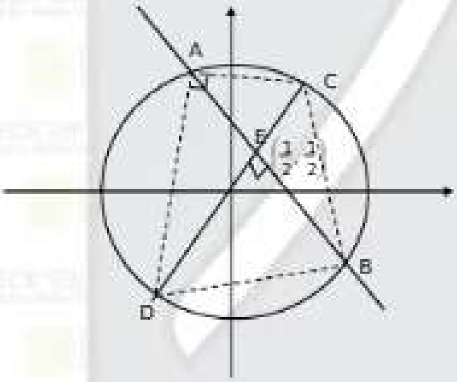
$$y\left(\frac{\pi}{6}\right) = \frac{1}{2\ell n\frac{\sqrt{3}}{2}} \Rightarrow y\left(\frac{\pi}{6}\right) = \frac{1}{\ell n3 - 2\ell n2}$$

15. The line $x + y = 1$ cut circle $x^2 + y^2 = 4$ at two points A and B. Another chord CD is perpendicular to AB and passing through mid-point of AB, then area of quadrilateral ADBC is –

(1) 2 (2) 7 (3) $2\sqrt{14}$ (4) $\sqrt{\dfrac{9+4\sqrt{2}}{2}}$

Ans. **(3)**

Sol. CE.ED = AE.EB



AB : $x + y = 1$

CD : $x - y = 0$

$\Rightarrow$ $E = \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$, $C(\sqrt{2}, \sqrt{2})$, $D(-\sqrt{2}, -\sqrt{2})$

$$ED = \sqrt{\frac{9+4\sqrt{2}}{2}},\ EC = \sqrt{\frac{9-4\sqrt{2}}{2}}$$

CE. ED = AE. AE $\Rightarrow AE^2 = \sqrt{\dfrac{81-32}{4}} = \dfrac{7}{2} \Rightarrow AE = \sqrt{\dfrac{7}{2}}$

Area $= \dfrac{1}{2}$ (CD) (AB) $= \dfrac{1}{2}(4)(2)\sqrt{\dfrac{7}{2}} = 2\sqrt{14}$