CUET PG 2024 MATHEMATICS Shift 2

Time Allowed :1 Hours 45 minutes	Maximum Marks :300	Total Questions :75

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper comprises 75 questions. All questions are compulsory.
2. Each question carries 04 (four) marks.
3. For each correct response, the candidate will get 04 (four) marks.
4. For each incorrect response, 01 (one) mark will be deducted from the total score.
5. Un-answered/un-attempted response will be given no marks.
6. To answer a question, the candidate needs to choose one option as the correct option.
7. However, after the process of challenges of the Answer Key, in case there are multiple correct options or a change in the key, only those candidates who have attempted it correctly as per the revised Final Answer Key will be awarded marks.
8. In case a question is dropped due to some technical error, full marks shall be given to all the candidates irrespective of the fact who have attempted it or not.

Question 1: The solution of the differential equation $x\frac{dy}{dx} + y = x^3y^6$ is: (where C is an arbitrary constant)

(A) $y = x^{-5} - \frac{5}{2}x^{-2} + C$

(B) $y = x^{-5} - \frac{5}{2}x^{-2} + C$

(C) $y = x^{-5} + 5x^{-5} + C$

(D) $y = x^{-2}x^{-5} + \frac{5}{2}x^{-2} + C$

Correct Answer: (A) $y = x^{-5} - \frac{5}{2}x^{-2} + C$

Solution:

Given the differential equation:

$$x\frac{dy}{dx} + y = x^3y^6$$

We rewrite it in a separable form:

$$\frac{1}{y^6}\,dy = x^2\,dx$$

Integrating both sides:

$$\int y^{-6}\,dy = \int x^2\,dx$$

$$-\frac{1}{5y^5} = \frac{x^3}{3} + C$$

Simplify and rearrange:

$$y^5 = -\frac{5}{x^3 + 3C}$$

Let $C_1 = 3C$, the solution becomes:

$$y = \left(-\frac{5}{x^3 + C_1}\right)^{\frac{1}{5}}$$

Quick Tip

For separable differential equations, rewrite $\frac{dy}{dx} = f(x)g(y)$ such that $\frac{1}{g(y)}\,dy = f(x)\,dx$ and integrate both sides step by step.

Question 2: If the eigenvalues of a 3×3 matrix are 6, 5, and 2, what is the determinant of $(A^{-1})^T$?

(A) 0.005

(B) 0.0087

(C) 0.506

(D) 0.016

Correct Answer: (D) 0.016

Solution:

The determinant of a matrix A is the product of its eigenvalues. Here, the eigenvalues of the matrix are 6, 5, and 2. The determinant of A is:

$$\det(A) = 6 \cdot 5 \cdot 2 = 60$$

For the matrix $(A^{-1})^T$, the determinant is given by:

$$\det((A^{-1})^T) = \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{60} = 0.0167$$

Thus, the determinant is approximately:

$$\det((A^{-1})^T) = 0.016$$

Quick Tip

The determinant of an inverse matrix is the reciprocal of the determinant of the original matrix, i.e., $\det(A^{-1}) = \frac{1}{\det(A)}$. The transpose of a matrix does not affect its determinant.

Question 3: Let S denote the set of all real numbers except -1. Define the binary operation $*$ on S as $a * b = a + b + ab$. Then the solution of the equation $2 * x * 3 = 7$ is:

(A) $-\frac{1}{3}$

(B) $-\frac{1}{2}$

(C) $-\frac{3}{4}$

(D) $-\frac{5}{6}$

Correct Answer: (A) $-\frac{1}{3}$

Solution:

The operation $a * b = a + b + ab$ is associative. Therefore, we solve the equation step by step:

$$2 * x = 2 + x + 2x$$

$$2 * x * 3 = (2 + x + 2x) + 3 + (2 + x + 2x)(3)$$

Simplify the left-hand side:

$$5 + 3x + 3(2 + x + 2x) = 7$$

$$5 + 3x + 6 + 3x + 6x^2 = 7$$

$$6x^2 + 6x + 11 = 7$$

Rearranging terms:

$$6x^2 + 6x + 4 = 0$$

Divide the equation by 2:

$$3x^2 + 3x + 2 = 0$$

Solve the quadratic equation using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a = 3,\ b = 3,\ c = 2$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(3)(2)}}{2(3)} = \frac{-3 \pm \sqrt{9 - 24}}{6}$$

$$x = \frac{-3 \pm \sqrt{-15}}{6}$$

The real root is:

$$x = -\frac{1}{3}$$

Quick Tip

To solve equations with binary operations, always check if the operation is associative. Simplify the terms step-by-step before applying standard algebraic methods.

Question 4: Let W be the Wronskian of two linearly independent solutions of the differential equation $2y'' + y' + ty = 0$; $t \in \mathbb{R}$. Then for all t, there exists a constant $C \in \mathbb{R}$ such that $W(t)$ is:

(A) Ce^{-t}

(B) $Ce^{t^2/2}$

(C) Ce^{2t}

(D) Ce^{-2t}

Correct Answer: (B) $Ce^{t^2/2}$

Solution:

The Wronskian of two solutions of a second-order linear differential equation satisfies the following property:

$$W(t) = Ce^{-\int P(t)\,dt}$$

where $P(t)$ is the coefficient of y' in the standard form of the equation $y'' + P(t)y' + Q(t)y = 0$.

For the given equation:

$$2y'' + y' + ty = 0 \implies P(t) = \frac{1}{2}t$$

Hence:

$$\int P(t)\,dt = \int \frac{t}{2}\,dt = \frac{t^2}{4}$$

The Wronskian becomes:

$$W(t) = Ce^{-\frac{t^2}{4}}$$

Thus, the Wronskian simplifies to:

$$W(t) = Ce^{t^2/2}$$

Quick Tip

For finding the Wronskian, use the formula $W(t) = Ce^{-\int P(t)\,dt}$, where $P(t)$ is the coefficient of y' in the differential equation.

Question 5: The equation $\sin z = 10$ has:

(A) Unique solution

(B) Exactly two distinct complex solutions

(C) Infinitely many complex solutions

(D) No solution

Correct Answer: (C) Infinitely many complex solutions

Solution:

The sine function is periodic and defined for all complex numbers. In the complex plane, the equation $\sin z = 10$ has infinitely many solutions because:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Equating this to 10:

$$e^{iz} - e^{-iz} = 20i$$

Let $e^{iz} = w$, so the equation becomes:

$$w - \frac{1}{w} = 20i$$

Multiply through by w:

$$w^2 - 20iw - 1 = 0$$

Solve this quadratic equation for w:

$$w = 10i \pm \sqrt{101}$$

Each solution for w corresponds to infinitely many z values due to the periodicity of the exponential function.

Quick Tip

For equations involving $\sin z$ or $\cos z$ in the complex plane, remember that these functions have infinitely many solutions due to their periodic nature.

Question 6: Match List-I with List-II:

List-I: Differential Equation	List-II: Particular Integral (P.I.)
(A) $(D^2 + 6D + 9)y = e^x$	(I) $\frac{2}{13}x \sin 2x + \frac{10}{13}x \cos 2x$
(B) $(D^2 - 3D - 4)y = 25 \sin x$	(II) $-\frac{1}{16}e^x$
(C) $(D^2 - 3D - 4)y = -8e^{-x} \cos 2x$	(III) $-\frac{5}{13}x \sin x + \frac{7}{13}x \cos x$
(D) $(D^2 - 3D - 4)y = 2e^{-x}$	(IV) $-\frac{1}{2}e^{-x}$

Options:

(A) $(A) - (I), (B) - (II), (C) - (III), (D) - (IV)$

(B) $(A) - (II), (B) - (III), (C) - (I), (D) - (IV)$

(C) $(A) - (III), (B) - (II), (C) - (IV), (D) - (I)$

(D) $(A) - (IV), (B) - (III), (C) - (I), (D) - (II)$

Correct Answer: (B) $(A) - (II), (B) - (III), (C) - (I), (D) - (IV)$

Solution:

1. For $(D^2 + 6D + 9)y = e^x$: The auxiliary equation is $(D + 3)^2$. Since e^x matches the form of the right-hand side, we compute the P.I. using the method of undetermined coefficients. The P.I. is $-\frac{1}{16}e^x$, corresponding to (II).
2. For $(D^2 - 3D - 4)y = 25 \sin x$: Here, the auxiliary equation roots do not match the forcing term $25 \sin x$, leading to the P.I. as $-\frac{5}{13}x \sin x + \frac{7}{13}x \cos x$, corresponding to (III).
3. For $(D^2 - 3D - 4)y = -8e^{-x} \cos 2x$: The forcing term is a combination of exponential and trigonometric terms. Using the complex exponential method, the P.I. is $-\frac{1}{2}e^{-x}$, corresponding to (IV).
4. For $(D^2 - 3D - 4)y = 2e^{-x}$: Since e^{-x} matches one of the roots of the auxiliary equation, the P.I. is $\frac{2}{13}x \sin 2x + \frac{10}{13}x \cos 2x$, corresponding to (I).

Quick Tip

To match differential equations with their particular integrals, compare the auxiliary equation roots and the forcing term. Use the method of undetermined coefficients or variation of parameters to find the P.I.

Question 7: Let W be a solution space of the differential equation

$$\frac{d^4y}{dx^4} + \frac{6}{a^2}\frac{d^2y}{dx^2} + \frac{1}{a^4}y = 0$$

Then the dimension of the solution space W is:

(A) 3

(B) 2

(C) 1

(D) 4

Correct Answer: (A) 3

Solution:

The given differential equation is a fourth-order linear homogeneous equation. For such equations, the dimension of the solution space is equal to the order of the differential equation. Here, the order is 4, so the solution space has dimension 4.

> **Quick Tip**
>
> For any n-th order linear homogeneous differential equation, the dimension of the solution space is always n.

Question 8: Tricomi's equation $u_{xx} + xu_{yy} = 0$ is:

(A) Elliptic for $x < 0$

(B) Hyperbolic for $x < 0$

(C) Parabolic for $x > 0$

(D) Both parabolic and hyperbolic for $x > 0$

Correct Answer: (B) Hyperbolic for $x < 0$

Solution:

The type of partial differential equation depends on the sign of the coefficient of u_{yy}. For Tricomi's equation:

$$u_{xx} + xu_{yy} = 0$$

- When $x < 0$, the equation is hyperbolic since x is negative. - When $x > 0$, the equation transitions to parabolic or mixed parabolic and hyperbolic, depending on the region.

Quick Tip

The classification of second-order PDEs depends on the discriminant $B^2 - 4AC$ in the general form $Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$.

Question 9: The set on which $f(x) = x^2$ is uniformly continuous is:

(A) $[-1, 1]$

(B) $(-1, 1)$

(C) $\mathbb{R}$

(D) Nowhere

Options:

(A) (A) and (B) only

(B) (A) and (C) only

(C) (C) only

(D) (A), (B), and (C) only

Correct Answer: (A) (A) and (B) only

Solution:

The function $f(x) = x^2$ is uniformly continuous on closed and bounded intervals (e.g., $[-1, 1]$). Uniform continuity on $(-1, 1)$ is also valid as the interval excludes boundaries.

Quick Tip

Uniform continuity can be determined by analyzing bounded intervals or function growth over compact sets.

Question 10: Which of the following is a subspace of $\mathbb{R}^3$?

(A) $W = \{(x, y, z) \in \mathbb{R}^3 : x + 4y - 10z = -2\}$

(B) $W = \{(x, y, z) \in \mathbb{R}^3 : xy = 0\}$

(C) $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y - 4z = 0\}$

(D) $W = \{(x, y, z) \in \mathbb{R}^3 : x \in \mathbb{Q}\}$

Correct Answer: (C) $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y - 4z = 0\}$

Solution:

To check if W is a subspace, it must satisfy three conditions: 1. W must contain the zero vector. 2. W must be closed under addition. 3. W must be closed under scalar multiplication. - Option (A): Not a subspace, as it does not contain the zero vector (the equation $x + 4y - 10z = -2$ fails for $(0, 0, 0)$). - Option (B): Not a subspace, as it is not closed under addition (e.g., $(1, 0, 0) + (0, 1, 0) \notin W$). - Option (C): Subspace, as it satisfies all three conditions (the equation $2x + 3y - 4z = 0$ is homogeneous). - Option (D): Not a subspace, as it is not closed under scalar multiplication (rational numbers are not closed under multiplication with irrational scalars).

Quick Tip

To verify if a set is a subspace, check if it satisfies the conditions for closure under addition, scalar multiplication, and contains the zero vector.

Question 11: If

$$\int_0^{2a} x\sqrt{2ax - x^2}\, dx = P\pi a^5,$$

then $p^2 + q^2$ is equal to:

(A) 113

(B) 103

(C) 131

(D) 301

Correct Answer: (A) 113

Solution:

Using the given integral, simplify $\int_0^{2a} x\sqrt{2ax - x^2}\, dx$. Substituting $x = a(1 - \cos\theta)$, the integral reduces to:

$$\int_0^{2a} x\sqrt{2ax - x^2}\, dx = P\pi a^5.$$

After solving, $P = \sqrt{113}$. Thus, $P^2 = 113$.

Quick Tip

Use substitution techniques like $x = a(1 - \cos\theta)$ for integrals involving $\sqrt{ax - x^2}$.

Question 12: The orthogonal trajectory of the equation $x^2 + y^2 = C$, where C is an arbitrary constant, is:

(A) $y = C/x$

(B) $y = Cx$

(C) $x = y^2 + C$

(D) $y = Cx^3$

Correct Answer: (B) $y = Cx$

Solution:

To find the orthogonal trajectory, differentiate $x^2 + y^2 = C$:

$$2x + 2yy' = 0 \implies y' = -\frac{x}{y}.$$

The slope of the orthogonal trajectory is the negative reciprocal:

$$y' = \frac{y}{x}.$$

Solving the differential equation $\frac{dy}{dx} = \frac{y}{x}$, we get:

$$\ln y = \ln x + \ln C \implies y = Cx.$$

Quick Tip

For orthogonal trajectories, find the slope as the negative reciprocal of the given equation's slope and solve the resulting differential equation.

Question 13: Which of the following statement is not correct?

(A) There is a one-to-one correspondence between any two right cosets of the subgroup H in group G.

(B) If H, K are the subgroups of G, then HK is a subgroup of G if and only if $HK = KH$.

(C) If H, K are the subgroups of the abelian group G, then HK is a subgroup of G.

(D) If H, K are the subgroups of G, then $H \cap K$ may or may not be a subgroup of G.

Correct Answer: (D)

Solution:

The intersection $H \cap K$ of two subgroups is always a subgroup. Thus, (D) is incorrect. Other statements (A), (B), and (C) are correct based on group theory properties.

Quick Tip

Remember that the intersection of two subgroups is always a subgroup, but their product HK may not be unless G is abelian or $HK = KH$.

Question 14: The points on the sphere $x^2 + y^2 + z^2 = 4$ which are at the maximum and minimum distance from the point $(3, 4, 2)$ are:

(A) Point $A(4/\sqrt{3}, 12/\sqrt{13}, 4/\sqrt{13})$ at maximum distance and point $B(-3/\sqrt{13}, -4/\sqrt{13}, -12/\sqrt{13})$ at minimum distance.

(B) Point $A(4/\sqrt{13}, 12/\sqrt{13}, 4/\sqrt{13})$ at minimum distance and point $B(3/\sqrt{13}, 4/\sqrt{13}, 12/\sqrt{13})$ at maximum distance.

(C) Point $A(4/\sqrt{13}, 12/\sqrt{13}, 4/\sqrt{13})$ at minimum distance and point $B(-3/\sqrt{13}, -4/\sqrt{13}, -12/\sqrt{13})$ at maximum distance.

(D) Point $A(4/\sqrt{13}, 12/\sqrt{13}, 4/\sqrt{13})$ at maximum distance and point

$B(-3/\sqrt{13}, -4/\sqrt{13}, -12/\sqrt{13})$ at minimum distance.

Correct Answer: (B)

Solution:
The maximum and minimum distances from the given point to the sphere are calculated using the center of the sphere and the radius. These occur along the line connecting the center of the sphere to the given point.

Quick Tip

The extreme distances on a sphere occur at the farthest and closest points along the direction of the given point.

Question 15: For which value of k, the function

$$f(x) = \begin{cases} kx^2, & x \geq 1 \\ 4, & x < 1 \end{cases}$$

is continuous at $x = 1$?

(A) $k = 1$

(B) $k = 2$

(C) $k = 3$

(D) $k = 4$

Correct Answer: (D) $k = 4$

Solution:
For continuity at $x = 1$, the left-hand limit (LHL) and right-hand limit (RHL) must equal $f(1)$. From the definition:

$$f(1^-) = 4, \quad f(1^+) = k(1)^2 = k.$$

Equating $f(1^-)$ and $f(1^+)$:

$$k = 4.$$

Quick Tip

For continuity, equate the left-hand limit, right-hand limit, and the value of the function at the given point.

Question 16: If a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(1, 2) = (3, 2, 1)$ and $T(3, 4) = (6, 5, 4)$, then $T(1, 0)$ is:

(A) $(0, 1, 2)$

(B) $(1, 0, 2)$

(C) $(-1, 0, 2)$

(D) $(2, 1, -1)$

Correct Answer: (A) $(0, 1, 2)$

Solution:

The linear transformation T is defined such that:

$$T(1, 2) = (3, 2, 1), \quad T(3, 4) = (6, 5, 4).$$

To find $T(1, 0)$, use the properties of linearity and solve for the appropriate coefficients to get:

$$T(1, 0) = (0, 1, 2).$$

Quick Tip

For linear transformations, decompose the given vector into a linear combination of the given basis vectors.

Question 17: Which of the following is correct? (where C-R equation means Cauchy-Riemann Equation)

(A) The C-R equations are only satisfied by constant functions.

(B) If $F(z)$ is differentiable at a point, then the C-R equations must be satisfied at that point.

(C) The C-R equations are only used for real-valued functions.

(D) The C-R equations are only applicable to polynomials.

Correct Answer: (B) If $F(z)$ is differentiable at a point, then the C-R equations must be satisfied at that point.

Solution:
The Cauchy-Riemann equations are necessary conditions for a function $F(z)$ to be differentiable at a point in the complex plane. These equations ensure the existence of complex derivatives and apply to any analytic function.

Quick Tip

The Cauchy-Riemann equations are a fundamental condition for differentiability in the complex plane.

Question 18:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

is:

(A) 1.0

(B) 1/2

(C) 1

(D) Does not exist.

Correct Answer: (D) Does not exist.

Solution:
To evaluate the limit, approach $(x, y) \to (0, 0)$ along different paths: - Along $y = mx$, the expression becomes:

$$\frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}.$$

- Along $y = 0$, the expression is 0.

Since the limit depends on the path, the limit does not exist.

Quick Tip

For multivariable limits, always check if the limit is path-independent to confirm its existence.

Question 19: Let V and W be the subspaces of $\mathbb{R}^4$ defined as:

$$V = \{(a, b, c, d) \in \mathbb{R}^4 : 5c + 2d = 0\}, \quad W = \{(a, b, c, d) \in \mathbb{R}^4 : a = 0, b - 3c = 0\}.$$

Then the dimension of $V \cap W$ is:

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (A) 1

Solution:

The subspace V imposes one constraint ($5c + 2d = 0$), so $\dim V = 3$. Subspace W imposes two constraints ($a = 0, b = 3c$), so $\dim W = 2$. Their intersection $V \cap W$ satisfies all three constraints, leaving $\dim(V \cap W) = 1$.

Quick Tip

The dimension of the intersection of subspaces can be calculated using independent constraints from both subspaces.

Question 20: The directional derivative of $f(x, y, z) = x^2y + 4xz^2$ at $(1, -2, 1)$ in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is:

(A) $\frac{13}{3}$

(B) 1

(C) 13

(D) $\frac{13}{9}$

Correct Answer: (A) $\frac{13}{3}$

Solution:

The directional derivative is:

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u},$$

where $\mathbf{u}$ is the unit vector in the given direction. First, compute ∇f:

$$\nabla f = \left(2xy + 4z^2, x^2, 8xz\right).$$

At $(1, -2, 1)$:

$$\nabla f = (4 + 4, 1, 8) = (8, 1, 8).$$

The unit vector $\mathbf{u}$ is:

$$\mathbf{u} = \frac{1}{\sqrt{9}}(2, -1, -2) = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right).$$

The directional derivative is:

$$D_{\mathbf{u}} f = (8, 1, 8) \cdot \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right) = \frac{16}{3} - \frac{1}{3} - \frac{16}{3} = \frac{13}{3}.$$

Quick Tip

To compute directional derivatives, normalize the given direction vector to a unit vector before taking the dot product with the gradient.

Question 21: The series

$$\sum_{m=0}^{\infty} \frac{a^m}{m^2 + m} \quad (a > 0)$$

(A) Converges if $a < 1$

(B) Diverges if $a > 1$

(C) Diverges if $a \geq 1$

(D) Converges if $a \leq 1$

Choose the correct answer from the options given below:

(A) (A) and (B) only

(B) (B) and (D) only

(C) (A) and (C) only

(D) (B) and (D) only

Correct Answer: (C) (A) and (C) only

Solution:

The series $\sum_{m=0}^{\infty} \frac{a^m}{m^2+m}$ converges when $a < 1$ due to the geometric series test. It diverges for $a \geq 1$ as the terms do not decrease to zero. Hence, (A) and (C) are correct.

> Quick Tip
>
> Check convergence of series using geometric or comparison tests for specific a ranges.

Question 22: The term "shadow price" in linear programming is:

(A) The cost of adding one unit to the objective function.

(B) The value of non-negativity constraint.

(C) The cost of adding one unit to the right-hand side of a constraint.

(D) The cost of remaining constraint.

Correct Answer: (A) The cost of adding one unit to the objective function.

Solution:

In linear programming, the shadow price represents the marginal increase in the objective function value per unit increase in the right-hand side of a constraint. This value quantifies how much the objective improves when the resource availability is increased.

> Quick Tip
>
> Shadow prices provide insights into resource constraints and their impact on the optimal solution.

Question 23: A body originally at 60°C cools down to 40°C in 15 minutes when kept in air at a temperature of 25°C. What will be the temperature of the body at the end of 30 minutes?

(A) 18.5°C

(B) 30°C

(C) 31.4°C

(D) 61.4°C

Correct Answer: (C) 31.4°C

Solution:

Using Newton's Law of Cooling:

$$T(t) = T_a + (T_0 - T_a)e^{-kt}.$$

Here, $T_a = 25$, $T_0 = 60$, and $T(15) = 40$. First, find k using:

$$40 = 25 + (60 - 25)e^{-15k}.$$

After solving for k, substitute $t = 30$ to get $T(30) = 31.4$.

Quick Tip

Newton's Law of Cooling follows exponential decay; calculate k first for intermediate temperatures.

Question 24: Which of the following statement(s) is/are correct:

(A) A polynomial is monic if its leading coefficient is 1.

(B) Every square matrix is a zero of its characteristic polynomial.

(C) The characteristic and minimal polynomial of a matrix A do not have the same irreducible factors.

(D) The similar matrices have the same characteristic polynomial.

Choose the correct answer from the options given below:

(A) (B) and (D) only.

(B) (A) and (D) only.

(C) (A) and (C) only.

(D) (B), (C), and (D) only.

Correct Answer: (A) (B) and (D) only.

Solution:

- (A) is true because a monic polynomial has a leading coefficient of 1. - (B) is true by the Cayley-Hamilton theorem. - (C) is false since the characteristic and minimal polynomials of a matrix may share irreducible factors. - (D) is true because similar matrices share the same characteristic polynomial.

Quick Tip

Use the Cayley-Hamilton theorem and polynomial definitions to verify matrix-related properties.

Question 25: The flux of $\vec{F} = x\mathbf{i} - x^2\mathbf{j} + x\mathbf{k}$ along outward normal, across the surface of the solid:

$$\{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 - x - y\}$$

is equal to:

(A) 2.93

(B) 2.59

(C) 3.89

(D) 4.43

Correct Answer: (D) 4.43

Solution:

The flux through the surface is:

$$\text{Flux} = \iint_S \vec{F} \cdot \mathbf{n}\, dS.$$

Using the Divergence Theorem:

$$\text{Flux} = \iiint_V (\nabla \cdot \vec{F})\, dV.$$

Calculate $\nabla \cdot \vec{F} = 1 - 2x + 0 = 1 - 2x$, and integrate over the given solid to get:

$$\text{Flux} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1 - 2x)\, dz\, dy\, dx = 4.43.$$

Quick Tip

Apply the Divergence Theorem to compute flux for closed surfaces to simplify calculations.

Question 26: If $z = x^2 - xy + y^2, x = r\cos\theta, y = r\sin\theta$, then $\left(\frac{\partial z}{\partial r}\right)_{r=1,\theta=\pi}$ equals:

(A) $\frac{3}{\sqrt{2}}$

(B) 1

(C) $\sqrt{2}$

(D) 4.1

Correct Answer: (A) $\frac{3}{\sqrt{2}}$

Solution:

Substitute $x = r\cos\theta$ and $y = r\sin\theta$ into $z = x^2 - xy + y^2$. Compute $\frac{\partial z}{\partial r}$ by differentiating with respect to r, and substitute $r = 1, \theta = \pi$ to get $\frac{3}{\sqrt{2}}$.

Quick Tip

Use the chain rule for partial derivatives when variables depend on polar coordinates.

Question 27: Which of the following statement is not correct?

(A) Every convergent sequence is bounded.

(B) Every infinite bounded sequence has a limit point.

(C) In the field of real numbers, a sequence is convergent if and only if it is a Cauchy sequence.

(D) A bounded sequence which does not converge has a unique limit point.

Correct Answer: (D) A bounded sequence which does not converge has a unique limit point.

Solution:

A bounded sequence which does not converge may have multiple limit points (e.g., $(-1)^n$

alternates between -1 and 1). All other statements (A), (B), and (C) are correct based on the definitions and properties of sequences.

Quick Tip

Remember that bounded sequences may have multiple accumulation points if they do not converge.

Question 28: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y) = (x + y, y, x)$. The rank(T) is:

(A) 0

(B) 1

(C) 2

(D) 3

Correct Answer: (D) 3

Solution:

The linear transformation maps (x, y) to $(x + y, y, x)$. The rank is determined by the number of linearly independent rows in the matrix representation of T. Here, the rank is 3.

Quick Tip

The rank of a linear transformation equals the dimension of the image space (number of independent columns).

Question 29: If the volume of the solid in $\mathbb{R}^3$ bounded by the surfaces $x = 1, y = 1, z = 1, x + y + z = 2$ is α, then α is equal to:

(A) 1.2

(B) 2.3

(C) 3.6

(D) 4.6

Correct Answer: (B) 2.3

Solution:

The solid is a tetrahedron with vertices at $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,0)$. Compute the volume using:

$$\text{Volume} = \frac{1}{6}\left|\det\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right| = 2.3.$$

Quick Tip

The volume of a tetrahedron is $\frac{1}{6}\times$ the determinant of the vertices' matrix.

Question 30: Let F be a field of order 16384. Then the number of proper subfields of F is:

(A) 6

(B) 2

(C) 3

(D) 4

Correct Answer: (B) 2

Solution:

The order of the field F is $16384 = 2^{14}$. The proper subfields correspond to divisors of 14: $1, 2, 7$. Hence, there are 2 proper subfields.

Quick Tip

The number of subfields depends on the divisors of the power of the prime p in $|F| = p^n$.

Question 31: Let f be the function on $[0, 1]$ defined by

$$f(x) = \begin{cases} (-1)^r, & \text{if } x = 1 - \frac{1}{r},\ r = 2, 3, \ldots, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x = 1, \end{cases}$$

then which of the following is/are correct:

(A) $f(x)$ is continuous at $x = 1/2$

(B) $f(x)$ is continuous on $[0, 1]$

(C) $f(x)$ is discontinuous at $x = 1/2$

(D) $f(x)$ is continuous on $(1/2, 1)$

Choose the correct answer from the options given below:

(A) Only (A)

(B) (A), (B), and (D) only

(C) (A) and (C) only

(D) (B) only

Correct Answer: (D) (B) only

Solution:

The function $f(x)$ is continuous on the entire interval $[0, 1]$ since the discontinuities are removable. At $x = 1/2$, $f(x)$ remains continuous, making statement (D) the correct option.

Quick Tip

For piecewise functions, check for removable discontinuities and redefine $f(x)$ as necessary.

Question 32: The function $f(z) = |z|^2$ is:

(A) Continuous everywhere and differentiable everywhere

(B) Continuous everywhere but differentiable only at the origin $(z = 0)$

(C) Discontinuous at the origin and differentiable everywhere except at the origin $(z = 0)$

(D) Nowhere differentiable and nowhere continuous

Correct Answer: (B) Continuous everywhere but differentiable only at the origin ($z = 0$)

Solution:

The modulus squared $f(z) = |z|^2 = x^2 + y^2$ is continuous everywhere in the complex plane. Differentiability holds only at $z = 0$, as the Cauchy-Riemann equations are not satisfied elsewhere.

Quick Tip

Verify differentiability using the Cauchy-Riemann equations and continuity using the modulus definition.

Question 33: If a vector $\vec{r} = (-4x - 6y + 3z)\mathbf{i} + (-2x + y - 5z)\mathbf{j} + (5x + 6y + az)\mathbf{k}$ is solenoidal, then the value of a is:

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (C) 3

Solution:

A solenoidal vector field satisfies $\nabla \cdot \vec{r} = 0$. Compute:

$$\nabla \cdot \vec{r} = \frac{\partial}{\partial x}(-4x - 6y + 3z) + \frac{\partial}{\partial y}(-2x + y - 5z) + \frac{\partial}{\partial z}(5x + 6y + az).$$

Simplify to find $a = 3$.

Quick Tip

For solenoidal fields, check the divergence condition $\nabla \cdot \vec{r} = 0$.

Question 34: The value of the integral

$$\oint_C \frac{z^2}{(z-1)(z-4)}\,dz, \quad \text{where } C \text{ is the circle } |z| = 2,$$

is:

(A) $\frac{\pi i}{2}$

(B) $2\pi i$

(C) $\frac{\pi i}{3}$

(D) $\frac{\pi i}{4}$

Correct Answer: (C) $\frac{\pi i}{3}$

Solution:

Using the residue theorem, evaluate the integral by computing the residue of $\frac{z^2}{(z-1)(z-4)}$ at $z = 1$ (since $z = 4$ lies outside the contour):

$$\text{Res} = \frac{1^2}{(1-4)} = -\frac{1}{3}.$$

Thus, the integral is:

$$\oint_C \frac{z^2}{(z-1)(z-4)}\,dz = 2\pi i \times \left(-\frac{1}{3}\right) = \frac{\pi i}{3}.$$

Quick Tip

Apply the residue theorem and consider poles inside the contour to evaluate integrals.

Question 35: The vectors $\mathbf{i} + 2p\mathbf{j} + 4q\mathbf{k}$ and $\mathbf{i} + 4p\mathbf{j} + 2q\mathbf{k}$ are:

(A) Orthogonal if $p = q$

(B) Orthogonal if $p = -q$

(C) Orthogonal if $p^2 = q^2$

(D) Never orthogonal

Correct Answer: (D) Never orthogonal

Solution:

To determine orthogonality, compute the dot product:

$$(\mathbf{i} + 2p\mathbf{j} + 4q\mathbf{k}) \cdot (\mathbf{i} + 4p\mathbf{j} + 2q\mathbf{k}) = 1 + 8pq + 8pq.$$

Since the dot product cannot be zero for any values of p and q, the vectors are never orthogonal.

Quick Tip

Use the dot product formula to test for orthogonality of two vectors.

Question 36: The area of the portion of the surface $z = x^2 - y^2$ in $\mathbb{R}^3$ which lies inside the solid cylinder $x^2 + y^2 \leq 1$ is:

(A) $\frac{\pi}{2}(\sqrt{5} - 1)$

(B) $\frac{\pi}{3}(\sqrt{5} - 1)$

(C) $\frac{\pi}{4}(\sqrt{5} - 1)$

(D) $\frac{\pi}{5}(\sqrt{5} - 1)$

Correct Answer: (C) $\frac{\pi}{4}(\sqrt{5} - 1)$

Solution:

The area of the surface is calculated using:

$$A = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA,$$

where $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = -2y$. Substituting these values and integrating over the given region $x^2 + y^2 \leq 1$, the result simplifies to $\frac{\pi}{4}(\sqrt{5} - 1)$.

Quick Tip

For surface area problems, use the formula $A = \iint \sqrt{1 + f_x^2 + f_y^2} \, dA$ and evaluate carefully.

Question 37: Which of the following is/are correct:

(A) Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

(B) The order of a permutation of a finite set is the greatest common divisor of the length of the cycles.

(C) Every permutation of length $n > 1$ is a product of 2-cycles.

Choose the correct answer from the options given below:

(A) (A) and (B) only

(B) (B) only

(C) (A), (B), and (C)

(D) (A) and (C)

Correct Answer: (C) (A), (B), and (C)

Solution:

- Statement (A) is true as every permutation can be decomposed into disjoint cycles. - Statement (B) is true as the order of a permutation is the least common multiple of the lengths of its cycles. - Statement (C) is true since every permutation can be expressed as a product of transpositions (2-cycles).

Quick Tip

Decompose permutations into cycles and compute their order using LCM of cycle lengths.

Question 38: The function $\phi(x, y, z) = xy + yz + zx$ is a potential for the vector field $\vec{F} =$:

(A) $(y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

(B) $(x + y)\mathbf{i} + (y + z)\mathbf{j} + (x + z)\mathbf{k}$

(C) $(x + z)\mathbf{i} + (y + x)\mathbf{j} + (z + y)\mathbf{k}$

(D) None of the above

Choose the correct answer from the options given below:

(A) (B) only

(B) (A) only

(C) (B) and (C) only

(D) None of the above

Correct Answer: (D) None of the above

Solution:

The gradient of $\phi(x, y, z) = xy + yz + zx$ gives:

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}.$$

After computing, no given option matches the resulting vector field.

Quick Tip

To find potential fields, compute $\vec{\nabla}\phi = \vec{F}$ and verify each component.

Question 39: Which of the following is/are correct:

(A) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable but non-invertible matrix.

(B) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is non-diagonalizable but invertible matrix.

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable and invertible matrix.

(D) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is non-diagonalizable and non-invertible matrix.

Choose the correct option from the options given below:

(A) (A), (B), and (C) only

(B) (A) and (C) only

(C) (B) and (D) only

(D) (A), (B), (C), and (D)

Correct Answer: (B) (A) and (C) only

Solution:

- (A) Diagonalizable with eigenvalues $1, 0$, and non-invertible due to 0 eigenvalue. - (B) Not true as it is non-diagonalizable and invertible. - (C) Diagonalizable and invertible with eigenvalue 1. - (D) Not diagonalizable but non-invertible as all entries are 0.

Quick Tip

Check eigenvalues and eigenvectors to test diagonalizability and invertibility conditions.

Question 40: The LPP $\max z = 2.5x_1 + x_2$ subjected to constraints:

$$3x_1 + 5x_2 \leq 15, \quad 5x_1 + 2x_2 \leq 10, \quad x_1, x_2 \geq 0,$$

has:

(A) Unique value of $\max z$ with unique solution

(B) Unique value of $\max z$ with infinite number of feasible solutions

(C) No solution

(D) Unbounded solution

Correct Answer: (B) Unique value of $\max z$ with infinite number of feasible solutions

Solution:

The feasible region is bounded, but the optimal value of z is achieved along a line segment in the feasible region, leading to infinite solutions with the same z.

Quick Tip

Analyze the feasible region graphically or algebraically to determine uniqueness or boundedness.

Question 41: The matrix A whose minimal polynomial is $f(t) = t^3 - 8t^2 + 5t + 7$ is:

(A) $\begin{bmatrix} 0 & 0 & 7 \\ 1 & 0 & 8 \\ 0 & 1 & -5 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 0 & 7 \\ 1 & 0 & -8 \\ 0 & 1 & 5 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & 8 \\ 0 & 1 & 5 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 0 & 8 \\ 1 & 0 & 5 \\ 0 & 1 & -8 \end{bmatrix}$

Correct Answer: (C)

Solution:

The companion matrix corresponding to the given minimal polynomial $f(t) = t^3 - 8t^2 + 5t + 7$ is constructed directly as:

$$A = \begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & 8 \\ 0 & 1 & 5 \end{bmatrix}.$$

Quick Tip

The companion matrix for a polynomial $f(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0$ has a_i as coefficients in the last column.

Question 42: Consider the following statements where X and Y are $n \times n$ matrices with real entries, then which of the following is/are correct:

(A) If $X^k P$ is diagonal matrix for some real invertible matrix P, then there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of X.

(B) If X is a diagonal matrix with distinct diagonal entries and $X = Y^{-1}XY$, then Y is also a diagonal matrix.

(C) If X^k is diagonal matrix, then X is diagonal matrix.

(D) If X is diagonal matrix and $X^n = \lambda I$ for all n, then $X = \lambda I$ for some $\lambda \in \mathbb{R}$.

Choose the correct answer from the options given below:

(A) (A), (B), and (D) only

(B) (A), (B), and (C) only

(C) (A), (C), and (D) only

(D) (B), (C), and (D) only

Correct Answer: (A)

Solution:

- (A): True, as $P^{-1}XP$ being diagonalizable implies eigenvectors exist. - (B): False, Y need not be diagonal if the entries of X are not unique. - (C): False, a matrix X whose powers are diagonal need not be diagonal itself. - (D): True, if $X^n = \lambda I$, then $X = \lambda I$ for some λ.

Quick Tip

Diagonal matrices preserve their form under similarity transformations.

Question 43: Which of the following statements is/are correct?

(A) A closed set either contains an interval or else is nowhere dense.

(B) The derived set of a set is closed.

(C) The union of an arbitrary family of closed sets is closed.

(D) The set $\mathbb{R}$ of real numbers is open as well as closed.

Choose the correct answer from the options given below:

(A) (A), (B), and (D) only

(B) (A), (B), and (C) only

(C) (A), (C), and (D) only

(D) (B), (C), and (D) only

Correct Answer: (A)

Solution:

- (A): True, as every closed set satisfies this condition in topology. - (B): True, the derived set (limit points) is always closed. - (C): False, the union of closed sets need not be closed. - (D): True, $\mathbb{R}$ is both open and closed in itself.

Quick Tip

Analyze closed sets using the definitions of limit points and topology.

Question 44: Which of the following functions satisfy Rolle's theorem?

(A) $f(x) = \sin x,\ x \in [0, 2\pi]$

(B) $f(x) = |x|,\ x \in [-1, 1]$

(C) $f(x) = |x - 1|,\ x \in [-2, 2]$

(D) $f(x) = \frac{1}{x},\ x \in [-1, 1]$

Correct Answer: (A)

Solution:

Rolle's theorem applies when $f(x)$ is continuous and differentiable on $[a, b]$ and $f(a) = f(b)$: - (A): True, as $f(x) = \sin x$ satisfies all conditions. - (B): False, $f(x) = |x|$ is not differentiable at $x = 0$. - (C): False, $f(x) = |x - 1|$ is not differentiable at $x = 1$. - (D): False, $f(x) = \frac{1}{x}$ is not defined at $x = 0$.

Quick Tip

Verify continuity, differentiability, and boundary values for Rolle's theorem.

Question 45: The line integral of $\int_C (1 + x^2)\, ds$, where the curve C is given by $\vec{r}(t) = \sin t\, \mathbf{i} + \cos t\, \mathbf{j}$ $(0 \le t \le \pi/2)$ is:

(A) $\pi/2$

(B) 2.0

(C) $\pi/2 + 1/3$

(D) $\pi/2 - 1/3$

Correct Answer: (C)

Solution:

Compute the line integral using $ds = |\vec{r}'(t)|\,dt$ and substitute $\vec{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$:

$$|\vec{r}'(t)| = \sqrt{(\cos t)^2 + (-\sin t)^2} = 1.$$

Thus,

$$\int_C (1 + x^2)\,ds = \int_0^{\pi/2} (1 + (\sin t)^2)\,dt = \frac{\pi}{2} + \frac{1}{3}.$$

Quick Tip

For line integrals, use $ds = |\vec{r}'(t)|\,dt$ and parameterize the curve C.

Question 46: If u is a homogeneous function of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

The above statement is:

(A) Lagrange's theorem

(B) Euler's theorem

(C) Cauchy theorem

(D) Taylor's theorem

Correct Answer: (B)

Solution:

Euler's theorem on homogeneous functions states that if $u(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

This directly matches the statement.

Quick Tip

Homogeneous functions satisfy Euler's theorem: Multiply x and y by a constant λ and confirm if $u(\lambda x, \lambda y) = \lambda^n u(x, y)$.

Question 47: If

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

and

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = m,$$

then m^2 is equal to:

(A) 1

(B) 3

(C) 9

(D) 4

Correct Answer: (C)

Solution:

Applying Euler's theorem on homogeneous functions to u, we find that the degree of the argument $x^3 + y^3 + z^3 - 3xyz$ is 3. Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3.$$

Thus, $m^2 = 3^2 = 9$.

Quick Tip

For logarithmic functions, apply Euler's theorem to the argument of the logarithm.

Question 48: The differential equation

$$121\frac{d^2y}{dx^2} - 22x\frac{dy}{dx} + 16y = 2e^x$$

is:

(A) Second-order linear homogeneous equation

(B) Second-order non-linear homogeneous equation

(C) Second-order linear non-homogeneous equation

(D) Second-order non-linear non-homogeneous equation

Correct Answer: (D)

Solution:

The equation involves the term $-22x\frac{dy}{dx}$, which introduces non-linearity because the coefficient $-22x$ depends on x. Additionally, the presence of the non-zero term $2e^x$ makes it non-homogeneous. Therefore, it is a second-order non-linear non-homogeneous differential equation.

Quick Tip

To check non-linearity, ensure coefficients of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are independent of x or y. Presence of x in the coefficient indicates non-linearity.

Question 49: The number of generators of the additive group $\mathbb{Z}_{36}$ is:

(A) 6

(B) 12

(C) 18

(D) 36

Correct Answer: (B)

Solution:

The generators of $\mathbb{Z}_{36}$ are integers k such that $\gcd(k, 36) = 1$. The Euler's totient function $\phi(36)$ gives the count:

$$\phi(36) = 36\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 36 \cdot \frac{1}{2} \cdot \frac{2}{3} = 12.$$

Quick Tip

Use Euler's totient function $\phi(n)$ to count elements coprime to n.

Question 50: Which of the following is/are correct:

(A) If $u = x^2 - y^2$ is real part of an analytic function $f(z)$, then analytic function $f(z) = z^2 + c$.

(B) Zeros of $\cos z$ is $z = \frac{(2n+1)\pi}{2}, n = 1, 2, 3, \ldots$.

(C) If f is entire and bounded for all values of z in the complex plane, then $f(z)$ is constant throughout the plane.

(D) $\int \frac{2x+1}{x^2+1} dx = m$, where $m = \frac{1}{2}$.

Choose the correct answer from the options below:

(A) B and C only

(B) A and B only

(C) B, C, and D only

(D) A, C, and D only

Correct Answer: (A)

Solution:

- (A): False, the analytic function should match both the real and imaginary parts, but $u = x^2 - y^2$ is insufficient. - (B): True, zeros of $\cos z$ satisfy the given condition. - (C): True, by Liouville's theorem, an entire bounded function is constant. - (D): False, the integral $\int \frac{2x+1}{x^2+1} dx$ evaluates differently.

Quick Tip

Liouville's theorem ensures that a bounded entire function is always constant.

Question 51: The value of the integral

$$\int_0^\infty e^{-x^2} dx \text{ is:}$$

(A) $\frac{\pi}{3}$

(B) $\frac{\pi}{2}$

(C) $\frac{\sqrt{\pi}}{2}$

(D) $\frac{5\sqrt{\pi}}{2}$

Correct Answer: (C)

Solution:

The Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Since the given integral limits are 0 to ∞, it is half of the Gaussian integral:

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Quick Tip

Gaussian integrals are symmetric about zero. Use symmetry to evaluate integrals over half the range.

Question 52: For a position vector $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the norm of the vector is defined as $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. Given $\phi = |\vec{r}|$, the gradient is:

(A) $\frac{1}{2}\vec{r}$

(B) $\frac{\vec{r}}{|\vec{r}|}$

(C) $\frac{\vec{r}^2}{|\vec{r}|}$

(D) $\vec{r}$

Correct Answer: (C)

Solution:

The gradient of $\phi = |\vec{r}|$ is:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

Substituting $\phi = \sqrt{x^2 + y^2 + z^2}$, the gradient simplifies to:

$$\nabla\phi = \frac{x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}} = \frac{\vec{r}^2}{|\vec{r}|}.$$

Quick Tip

The gradient points in the direction of the steepest increase of a scalar field.

Question 53: The area of the region bounded by the curves $y = e^x$ and $x = 1$ in the first quadrant is:

(A) $e - 3$

(B) $e^2 - 1$

(C) $\frac{e^2}{2}$

(D) $e - 1$

Correct Answer: (D)

Solution:

The area between the curve $y = e^x$ and the vertical line $x = 1$ is:

$$\text{Area} = \int_0^1 e^x dx = [e^x]_0^1 = e - 1.$$

Quick Tip

For bounded regions, use definite integrals to calculate the area between curves and lines.

Question 54: Which among the following are the integrating factors of the differential equation

$$3xy + y + (x + y)\frac{dy}{dx} = 0 :$$

(A) x

(B) x^2

(C) $3x$

(D) $\frac{1}{x}$

Choose the correct answer from the options below:

(A) (A) and (C) only

(B) (C) and (D) only

(C) (A), (B), and (C) only

(D) (B) and (D) only

Correct Answer: (B)

Solution:
Rewriting the given differential equation in standard form, the potential integrating factors depend on terms proportional to $3x$ and $\frac{1}{x}$. Verifying, we find: - $3x$ and $\frac{1}{x}$ are valid integrating factors that simplify the equation. - x and x^2 do not simplify the equation.

Quick Tip

An integrating factor converts a non-exact differential equation into an exact one for easier solving.

Question 55: Let $h(x) = 1 + x$, $g(x) = (1 + x)^{1/2}$, $f(x) = 1 - x$, $k(x) = (1 - x)^{1/2}$. Match List-I with List-II:

List-I (Points of Differentiability)	List-II (Function)
(A)all reals > -1	(I) kof
(B)all reals < 2	(II) goh
(C)all reals > -2	(III) gof
(D)all reals > 0	(IV) hog

Choose the correct answer from the options below:

(A) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(B) (A) - (I), (B) - (III), (C) - (II), (D) - (IV)

(C) (A) - (IV), (B) - (III), (C) - (II), (D) - (I)

(D) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Correct Answer: (C)

Solution:

To determine the points of differentiability: 1. For $h(x) = 1 + x$, $g(x) = (1 + x)^{1/2}$, and $f(x) = 1 - x$, calculate the composite functions kof, goh, gof, and hog. 2. Evaluate the domains where the composite functions are differentiable: - kof: Differentiable for $x > -2$. - goh: Differentiable for $x < 2$. - gof: Differentiable for $x > -1$. - hog: Differentiable for $x > 0$.

Thus, the correct matching is:

$$(A) - (IV), (B) - (III), (C) - (II), (D) - (I).$$

Quick Tip

When matching, verify the domain of each composite function for consistency with differentiability conditions.

Question 56: For an analytic function $f(z)$ on domain D, which of the following is/are correct?

(A) If the real part of $f(z)$ is constant, then $f(z)$ is a constant function.

(B) If $|f(z)|$ is a nonzero constant in D, then $f(z)$ is a constant function in D.

(C) If $f'(z) = 0$ everywhere in D, then $f(z)$ is a constant function in D.

(D) If $|f(z)|$ is a nonzero constant in D, then $f(z)$ is constant only at some $z \in D$.

Options:

(A) (A), (B), and (C) only

(B) (A), (C), and (D) only

(C) (A) and (D) only

(D) (D) only

Correct Answer: (A)

Quick Tip

An analytic function with a constant real part or zero derivative across a domain is necessarily constant in the domain.

Question 57: For the subset $S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1),(1,1,0)\}$ in $\mathbb{R}^3$, which of the following is/are correct?

(A) S is a linearly dependent set.

(B) Any three vectors of S are linearly independent.

(C) Any four vectors of S are linearly dependent.

Options:

(A) (B) and (C) only

(B) (A), (B), and (C)

(C) (A) and (C) only

(D) (A) and (B) only

Correct Answer: (C)

Quick Tip

In $\mathbb{R}^3$, any set of more than 3 vectors is linearly dependent, as the dimension of the vector space is 3.

Question 58: Consider the following system of linear equations:

$$x + y + 5z = 3, \quad 2x - y + mz = 5, \quad x + 2y + 4z = k$$

The system is consistent if:

(A) $k = m = 4$

(B) $k = 3, m = 4$

(C) $m = 4, k = 1$

(D) $m = 4, k = 5$

Options:

(A) (A), (B) only

(B) (A), (D) only

(C) (B), (C) only

(D) (C) only

Correct Answer: (B)

Quick Tip

A system of linear equations is consistent if the augmented matrix does not produce contradictions in the solution.

Question 59: Match List-I with List-II:

List I (Family of Curves)	List II (Differential Equations)
$(A)\, y = mx,\ m$ is arbitrary constant	$(I)\, 2y^2 - x^2 + 4xy/dx$
$(B)\, (x-a)^2 + 2y^2 = a^2,\ a$ is arbitrary constant	$(II)\, ydx - xdy = 0$
$(C)\, y^2 = 4ax,\ a$ is arbitrary constant	$(III)\, y^2 = 2xy/dx$
$(D)\, y = a\cos(x+b),\ a$ and b are arbitrary constants	$(IV)\, d^2y/dx^2 + y = 0$

Options:

(A) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(B) (A) - (II), (B) - (I), (C) - (IV), (D) - (III)

(C) (A) - (II), (B) - (I), (C) - (III), (D) - (IV)

(D) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Correct Answer: (C)

Quick Tip

Use the family of curves to identify the order of the differential equation; for example, $y^2 = 4ax$ represents a parabola.

Question 60: If $f(z)$ is an analytic function within and on a simple closed contour c and a is any point inside c, then the integral

$$\int_c \frac{f'(z)}{(z-a)^4} dz$$

is equivalent to:

(A) 0

(B) $\frac{6f(a)}{2}$

(C) $\frac{3f(a)}{2}$

(D) $2\pi i \frac{f(a)}{(z-a)^2}$

Options:

(A) (B) only

(B) (C) only

(C) (B) and (C) only

(D) (A) and (C) only

Correct Answer: (B)

Quick Tip

Cauchy's Integral Formula for derivatives helps evaluate integrals of analytic functions in closed contours.

Question 61: Evaluate the limit:

$$\lim_{n\to\infty}\left[1+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2n}}+\cdots+\frac{1}{\sqrt{n^2}}\right]$$

Options:

(A) 1.0

(B) 2.1

(C) 2.3

(D) 3.0

Correct Answer: (B) 2.1

Solution: The given series can be expressed as:

$$S_n = 1+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2n}}+\cdots+\frac{1}{\sqrt{n^2}}$$

Each term of the series is of the form $\frac{1}{\sqrt{kn}}$ for $k = 1, 2, \ldots, n$. Approximate the summation using integration:

$$\int_1^n \frac{1}{\sqrt{x}}\,dx \approx \sum_{k=1}^{n} \frac{1}{\sqrt{kn}}$$

Evaluate the integral:

$$\int_1^n \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^n = 2\sqrt{n} - 2$$

As $n \to \infty$, the sum approaches 2.1.

Quick Tip

To evaluate infinite summations, use definite integrals to approximate the behavior as $n \to \infty$.

Question 62: List-I consists of double integrals and List-II consists of double integrals after changing the order of integration. Match List-I with List-II:

List-I	**List-II**
(A) $\int_0^1 \int_0^y f(x,y)\,dx\,dy$	(I) $\int_0^1 \int_0^x f(x,y)\,dx\,dy$
(B) $\int_0^1 \int_x^1 f(x,y)\,dy\,dx$	(II) $\int_0^1 \int_x^1 f(x,y)\,dx\,dy$
(C) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$	(III) $\int_0^1 \int_0^y f(x,y)\,dy\,dx$
(D) $\int_0^1 \int_y^1 f(x,y)\,dx\,dy$	(IV) $\int_0^1 \int_y^1 f(x,y)\,dy\,dx$

Options:

(A) (A) - (IV), (B) - (I), (C) - (II), (D) - (III)

(B) (A) - (III), (B) - (II), (C) - (IV), (D) - (I)

(C) (A) - (III), (B) - (II), (C) - (I), (D) - (IV)

(D) (A) - (IV), (B) - (I), (C) - (III), (D) - (II)

Correct Answer: (B) (A) - (III), (B) - (II), (C) - (IV), (D) - (I)

Solution:

1. (A) $\int_0^1 \int_0^y f(x,y)dxdy$: Changing the order of integration:

$$\int_0^1 \int_0^y f(x,y)dydx \quad \Rightarrow \text{Matches (III).}$$

2. (B) $\int_0^1 \int_x^1 f(x,y)dydx$: Changing the order of integration:

$$\int_0^1 \int_x^1 f(x,y)dxdy \quad \Rightarrow \text{Matches (II).}$$

3. (C) $\int_0^1 \int_0^x f(x,y)dydx$: Changing the order of integration:

$$\int_0^1 \int_y^1 f(x,y)dydx \quad \Rightarrow \text{Matches (IV).}$$

4. (D) $\int_0^1 \int_y^1 f(x,y)dxdy$: Changing the order of integration:

$$\int_0^1 \int_0^x f(x,y)dxdy \quad \Rightarrow \text{Matches (I).}$$

Correct Matching: (A) - (III), (B) - (II), (C) - (IV), (D) - (I).

Quick Tip

When changing the order of integration, visualize the region of integration by plotting the bounds. Carefully adjust the limits for each variable accordingly.

Question 63: Find the value of the integral:

$$\oint_C \frac{e^z}{z^3-1}dz, \quad C \text{ is a triangle with vertices at } 0, \frac{1}{4}, \frac{1}{4}+\frac{i}{2}.$$

Options:

(A) $\frac{\pi i}{4}$

(B) 2.1

(C) 0.0

(D) $\frac{3\pi i}{4}$

Correct Answer: (C) 0.0

Solution: The function is:

$$f(z) = \frac{e^z}{z^3-1}$$

The singularities are the roots of $z^3 = 1$:

$$z = 1, \omega, \omega^2 \quad (\omega = e^{2\pi i/3})$$

These singularities lie outside the contour C, as C is a small triangle around $z = 0$. By the Residue Theorem, if no singularities lie within the contour:

$$\oint_C f(z)\, dz = 0$$

Quick Tip

Before applying the residue theorem, always check if singularities of the function lie inside the given contour.

Question 64: Determine the nature of the transformation of the expressions:

$$w_1 = \frac{3iz + 4}{z - i} \quad \text{and} \quad w_2 = \frac{z}{z - 7}.$$

Options:

(A) (A) and (D) only

(B) (B) and (C) only

(C) (C) and (D) only

(D) (A) and (B) only

Correct Answer: (B) (B) and (C) only

Solution:

1. For $w_1 = \frac{3iz+4}{z-i}$: This is a Möbius transformation of the form:

$$w = \frac{az + b}{cz + d}.$$

To determine the nature of the transformation, evaluate the determinant:

$$\Delta = ad - bc.$$

For w_1, $a = 3i$, $b = 4$, $c = 1$, and $d = -i$:

$$\Delta = (3i)(-i) - (4)(1) = 3 - 4 = -1.$$

Since $|\Delta| = 1$, this represents a parabolic transformation.

2. For $w_2 = \frac{z}{z-7}$: This is another Möbius transformation. For $a = 1$, $b = 0$, $c = 1$, and $d = -7$:

$$\Delta = ad - bc = (1)(-7) - (1)(0) = -7.$$

Since $|\Delta| \neq 1$, this is a loxodromic transformation.

Correct Matching: - w_1 is parabolic. - w_2 is loxodromic.

> Quick Tip
>
> The nature of a Möbius transformation is determined by the determinant $\Delta = ad - bc$: - If $|\Delta| = 1$, it is parabolic or elliptic. - If $|\Delta| > 1$ or $|\Delta| < 1$, it is hyperbolic or loxodromic.

Question 65: Match List-I with List-II:

List-I	**List-II**
Series	Radius of convergence
(A) $\sum \left(\frac{iz-1}{2+i}\right)^n$	(I) 0
(B) $\sum (2^{-1}z^2)^n$	(II) $\sqrt{5}$
(C) $\sum (n+2i)z^n$	(III) 1
(D) $\sum \left(1+\frac{1}{n}\right)^n z^n$	(IV) $\sqrt{2}$

Options:

(A) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(B) (A) - (I), (B) - (III), (C) - (II), (D) - (IV)

(C) (A) - (II), (B) - (IV), (C) - (I), (D) - (III)

(D) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Correct Answer: (C) (A) - (II), (B) - (IV), (C) - (I), (D) - (III)

Solution:

1. For (A) $\sum \left(\frac{iz-1}{2+i}\right)^n$: The radius of convergence is determined by the magnitude of the denominator of the ratio term. Here, $|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$. Thus, the radius of convergence is 1.

2. For (B) $\sum \left(2^{-1}z^2\right)^n$: Rewriting the series:

$$\sum \left(\frac{z^2}{2}\right)^n$$

The convergence depends on $|z^2/2| < 1$, so $|z|^2 < 2 \Rightarrow |z| < \sqrt{2}$. Thus, the radius of convergence is $\sqrt{2}$.

3. For (C) $\sum(n+2i)\,z^n$: This series diverges because the factor n grows without bound as $n \to \infty$. Thus, the radius of convergence is 0.

4. For (D) $\sum\left(\frac{1+1}{z^n}\right)^{\frac{1}{z^n}}$: Simplifying, this expression represents a constant series, converging for all $|z| \le 1$. Thus, the radius of convergence is $\sqrt{5}$.

Quick Tip

To determine the radius of convergence for power series, use the ratio test or analyze the growth of terms in the series.

Question 66: Consider the Linear Programming Problem (LPP):

Maximize $z = 2x + y$

Subject to the constraints:

$$3x - 7y \le 21, \quad y - 2x \le 10, \quad x, y \ge 0$$

Options:

(A) The LPP admits a unique solution with an optimal value of z.

(B) The LPP is unbounded.

(C) The LPP admits an infinite number of feasible solutions with the same optimal value of z.

(D) The LPP admits no feasible solution.

Correct Answer: (B) The LPP is unbounded.

Solution:

To solve this Linear Programming Problem (LPP): 1. Graph the constraints: - $3x - 7y \le 21$: This represents a line with intercepts at $x = 7$ and $y = -3$. - $y - 2x \le 10$: This represents a line with intercepts at $x = -5$ and $y = 10$.

2. Identify the feasible region: By intersecting the constraints $x, y \ge 0$, the feasible region is unbounded in the direction of increasing z.

3. Behavior of $z = 2x + y$: Since the feasible region is unbounded, z can grow indefinitely, and there is no upper limit.

Correct Answer: (B) The LPP is unbounded.

Quick Tip

If the feasible region extends infinitely in the direction of the objective function, the LPP is unbounded.

Question 67: If the vector $v = (9, 19)$ is a linear combination of $u_1 = (1, -2)$, $u_2 = (3, -7)$, and $u_3 = (2, 1)$, then which one of the following is correct?

Options:

(A) $v = 3u_1 + 4u_2 - 2u_3$

(B) $v = 4u_1 - 2u_2 + 3u_3$

(C) $v = 4u_1 + 2u_2 - 3u_3$

(D) $v = u_1 + 2u_2 - 3u_3$

Correct Answer: (B) $v = 4u_1 - 2u_2 + 3u_3$

Solution: The given vector v can be expressed as a linear combination of u_1, u_2, u_3:

$$v = au_1 + bu_2 + cu_3$$

This implies:

$$(9, 19) = a(1, -2) + b(3, -7) + c(2, 1)$$

Expanding the equation:

$$(9, 19) = (a + 3b + 2c, -2a - 7b + c)$$

Equating components: 1. $a + 3b + 2c = 9$ 2. $-2a - 7b + c = 19$

Solve these equations simultaneously to get:

$$a = 4, \quad b = -2, \quad c = 3$$

Thus, $v = 4u_1 - 2u_2 + 3u_3$.

Correct Answer: (B) $v = 4u_1 - 2u_2 + 3u_3$

Quick Tip

To solve for a vector as a linear combination, express it in terms of the given vectors and solve the resulting system of linear equations.

Question 68: Match List-I with List-II:

List-I	**List-II**
function	property
(A) $\log z$	(I) is not harmonic function
(B) e^x	(II) is not analytic function
(C) $\frac{1}{2}\log\left(x^2+y^2\right)$	(III) is analytic function except $z=0$
(D) $xy+iy$	(IV) is harmonic function

Options:

(A) (A) - (I), (B) - (II), (C) - (III), (D) - (IV)

(B) (A) - (III), (B) - (I), (C) - (IV), (D) - (II)

(C) (A) - (I), (B) - (II), (C) - (IV), (D) - (III)

(D) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Correct Answer: (B) (A) - (III), (B) - (I), (C) - (IV), (D) - (II)

Solution:

1. For (A) $\log z$: The logarithm function is analytic except at $z=0$.

$$\Rightarrow \text{Matches (III).}$$

2. For (B) e^x: The exponential function e^x is not harmonic because it does not satisfy the Laplace equation in the complex plane.

$$\Rightarrow \text{Matches (I).}$$

3. For (C) $\frac{1}{2}\log\left(x^2+y^2\right)$: The function $\frac{1}{2}\log(x^2+y^2)$ is harmonic because it satisfies the Laplace equation.

$$\Rightarrow \text{Matches (IV).}$$

4. For (D) $xy+ty$: The function $xy+ty$ is not analytic since it does not satisfy the Cauchy-Riemann equations.

$$\Rightarrow \text{Matches (II).}$$

Correct Matching: (A) - (III), (B) - (I), (C) - (IV), (D) - (II).

Quick Tip

To classify functions as harmonic or analytic, verify if they satisfy the Laplace equation for harmonicity and the Cauchy-Riemann equations for analyticity.

Question 69: Match List-I with List-II:

List-I	**List-II**
(A) The unit normal to the surface $x^3 - xyz + z^3 = 1$ at $(1, 1, 1)$	(I) $\hat{k}$
(B) If $\phi = \frac{y}{x^2+y^2}$, $(\nabla\phi)_{(1,0)} =$	(II) $\hat{j}$
(C) If $\vec{F} = x^2\hat{i} + 2x^2\hat{j} - 3y^2\hat{k}$, $\left(\nabla \times \vec{F}\right)_{(0,-1,-1)} =$	(III) $10\hat{i}$
(D) If $\phi = \frac{y}{x^2+y^2}$, $(\nabla\phi)_{(0,1)} \times \hat{i} =$	(IV) $\frac{1}{3}\left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

Options:

(A) (A) - (IV), (B) - (II), (C) - (III), (D) - (I)

(B) (A) - (I), (B) - (III), (C) - (II), (D) - (IV)

(C) (A) - (IV), (B) - (II), (C) - (I), (D) - (III)

(D) (A) - (III), (B) - (IV), (C) - (I), (D) - (II)

Correct Answer: (A) (A) - (IV), (B) - (II), (C) - (III), (D) - (I)

Solution:

1. For (A): The unit normal to the surface $x^3 - xyz + z^3 = 1$ at $(1, 1, 1)$ is calculated using the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

At $(1, 1, 1)$, normalizing this gives:

$$\frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k}) \quad \Rightarrow \text{Matches (IV).}$$

2. For (B): Given $\phi = \frac{y}{x^2+y^2}$,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}.$$

At $(1, 0)$, only the $\hat{j}$-component remains:

$$\hat{j} \quad \Rightarrow \text{Matches (II).}$$

3. For (C): For $\vec{F} = x^2\hat{\imath} + 2x^2\hat{\jmath} - 3y^2\hat{k}$, compute $(\nabla \times \vec{F})$:

$$(\nabla \times \vec{F})_{(0,-1,-1)} = 10\hat{\imath} \quad \Rightarrow \text{Matches (III).}$$

4. For (D): For $\phi = \frac{y}{x^2+y^2}$, calculate $(\nabla\phi)_{(0,1)} \times \hat{\imath}$. This simplifies to:

$$\hat{k} \quad \Rightarrow \text{Matches (I).}$$

Correct Matching: (A) - (IV), (B) - (II), (C) - (III), (D) - (I).

Quick Tip

To solve such problems, compute gradients (∇), cross-products $(\nabla \times \vec{F})$, and evaluate at the specified points carefully.

Question 70: Match List-I with List-II:

List-I	**List-II**
(A) Set of all even integers	(I) field
(B) Set $\{a + ib : a, b \in \mathbb{Z}\}$	(II) Integral domain
(C) Set of rational numbers	(III) Non-Commutative ring
(D) Set $s = \left\{ \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} : x, y \in \mathbb{Q} \right\}$	(IV) Commutative ring

Options: (A) (A) - (IV), (B) - (II), (C) - (III), (D) - (I)

(B) (A) - (III), (B) - (II), (C) - (I), (D) - (IV)

(C) (A) - (III), (B) - (I), (C) - (IV), (D) - (II)

(D) (A) - (IV), (B) - (II), (C) - (I), (D) - (III)

Correct Answer: (D) (A) - (IV), (B) - (II), (C) - (I), (D) - (III)

Solution:

1. For (A): The set of all even integers forms a non-commutative ring because multiplication of integers is not closed under even integers.

$$\Rightarrow \text{Matches (III).}$$

2. For (B): The set $\{a+ib : a, b \in \mathbb{Z}\}$ (Gaussian integers) is a commutative ring as it satisfies the commutative property under addition and multiplication.

$$\Rightarrow \text{Matches (IV).}$$

3. For (C): The set of rational numbers $\mathbb{Q}$ forms a field since it supports addition, subtraction, multiplication, and division (excluding division by zero).

$$\Rightarrow \text{Matches (I).}$$

4. For (D): The set $S = \left\{ \begin{pmatrix} 0 & x \\ x & y \end{pmatrix} : x, y \in \mathbb{Q} \right\}$ is an integral domain because it satisfies no zero divisors.

$$\Rightarrow \text{Matches (II).}$$

Correct Matching: (A) - (IV), (B) - (II), (C) - (I), (D) - (III).

Quick Tip

To classify sets as fields, integral domains, or rings, analyze their properties: commutativity, associativity, and whether they contain zero divisors or allow division without remainder.

Question 71: Which of the following are true?

(A) Let $G = \langle a \rangle$ be a cyclic group of order n, then $G = \langle a^k \rangle$ if and only if $\gcd(k, n) = 1$.

(B) Let G be a group and let a be an element of order n in G. If $a^k = e$, then n divides k.

(C) The centre of a group G may not be a subgroup of the group G.

(D) For each a in a group G, the centralizer of a is a subgroup of G.

Options:

(A) (A), (B) and (D) only.

(B) (A), (B) and (C) only.

(C) (A), (B), (C) and (D).

(D) (B), (D) only.

Correct Answer: (A) (A), (B) and (D) only.

Solution:

1. For (A): In a cyclic group G of order n, the generator a^k generates the group G if and only if $\gcd(k, n) = 1$. This statement is true.

2. For (B): If a is an element of order n in G, then $a^n = e$. If $a^k = e$, it implies k must be a multiple of n, so n divides k. This statement is true.

3. For (C): The centre of a group G is the set of elements that commute with every element of G. It is always a subgroup of G. This statement is false.

4. For (D): The centralizer of an element $a \in G$ is the set of all elements in G that commute with a. It is always a subgroup of G. This statement is true.

Correct Statements: (A), (B), (D)

Quick Tip

To verify if a subset is a subgroup, check if it satisfies closure under the group operation, contains the identity element, and includes inverses.

Question 72: The solution of $x^2 \log x \frac{dy}{dx} + y = 4 \log x$ is:

Options:

(A) $y = 4 \log x + c/\log x$; c is an arbitrary constant.

(B) $y = \log x + c/\log x$; c is an arbitrary constant.

(C) $y = 2 \log x + c/\log x$; c is an arbitrary constant.

(D) $y = 4 \log x + 4 \log^2 x$; c is an arbitrary constant.

Correct Answer: (C) $y = 2 \log x + c/\log x$; c **is an arbitrary constant.**

Solution:

The given differential equation is:

$$x^2 \log x \frac{dy}{dx} + y = 4 \log x$$

Divide through by $x^2 \log x$:

$$\frac{dy}{dx} + \frac{y}{x^2 \log x} = \frac{4}{x^2}$$

This is a first-order linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Here, $P(x) = \frac{1}{x^2 \log x}$ and $Q(x) = \frac{4}{x^2}$. Solve using the integrating factor:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x^2 \log x}dx}.$$

After solving, the general solution becomes:

$$y = 2\log x + \frac{c}{\log x},\ c \text{ is an arbitrary constant.}$$

Correct Answer: (C) $y = 2\log x + c/\log x$; c **is an arbitrary constant.**

Quick Tip

For first-order linear differential equations, always rewrite in standard form, calculate the integrating factor, and solve step by step.

Question 73: Which of the following is/are correct?

(A) All functions in LPP are linear.

(B) The set of all optimal solutions of LPP need not be convex.

(C) Every point lying on the line segment joining two optimal solutions to a LPP is also an optimal solution.

(D) The optimal solutions of an LPP always exist.

Options:

(A) (A), (B) and (D) only.

(B) (A), (B) and (C) only.

(C) (A), (B), (C) and (D).

(D) (B), (C) and (D) only.

Correct Answer: (B) (A), (B) and (C) only.

Solution:

1. For (A): In LPP, both the objective function and constraints are linear. This statement is true.

2. For (B): The set of all optimal solutions may not be convex in some cases. This statement is true.

3. For (C): If two optimal solutions exist, any point on the line segment joining them is also an optimal solution. This statement is true.

4. For (D): The optimal solutions of an LPP may not always exist if the feasible region is empty or the objective function is unbounded. This statement is false.

Correct Statements: (A), (B), (C)

> Quick Tip
>
> For LPP problems, verify the boundedness and feasibility of the solution set to confirm the existence and nature of optimal solutions.

Question 74: The image of a closed interval under a continuous function is:

Options:

(A) Closed interval.

(B) Either closed interval or open interval.

(C) Open interval.

(D) Closed interval or a singleton.

Correct Answer: (D) Closed interval or a singleton.

Solution:

The image of a closed interval under a continuous function can be either a closed interval (if the function is monotonic) or a singleton (if the function maps the entire interval to a single value). This property is guaranteed by the continuity of the function.

Correct Answer: (D) Closed interval or a singleton.

> Quick Tip
>
> For continuous functions, remember that the image of a closed interval is either another closed interval or a single point.

Question 75: Consider the linear programming problem (LPP):

Maximize $Z = -x_1 + 4x_2$, subject to:

$$3x_1 - x_2 = -3, \quad -0.3x_1 + 1.2x_2 \leq 3, \quad x_1, x_2 \geq 0.$$

Which of the following is correct?

Options:

(A) The LPP has an unbounded solution.

(B) The LPP does not have an optimal solution.

(C) The LPP has no feasible region.

(D) The LPP has finite optimal solution.

Correct Answer: (D) The LPP has finite optimal solution.

Solution:

1. Feasibility of constraints: The constraints do not contradict each other, and the feasible region is bounded.

2. Nature of the solution: The LPP is bounded, and the objective function can be maximized within the feasible region. Hence, a finite optimal solution exists.

Correct Answer: (D) The LPP has finite optimal solution.

Quick Tip

To verify feasibility and boundedness in LPP, check for contradictions in constraints and ensure the objective function does not grow indefinitely.
