Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section A** contains a total of 30 **Multiple Choice Questions (MCQ).** Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section B** contains a total of 10 **Multiple Select Questions (MSQ).** Each MSQ type question is similar to MCQ but with a difference that there will be **one or more than one** choices that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1-mark questions, 1/3 marks will be deducted for each wrong answer. For all 2-mark questions, 2/3 marks will be deducted for each wrong answer. In **Section B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. A Scribble Pad will be provided for rough work.

Special Instructions / Useful Data

$\mathbb{N}$	The set of positive integers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$
$\ln x$	Natural logarithm of x
$\mathbb{P}(E)$	Probability of event E
$\mathbb{P}(E \mid F)$	Conditional probability of event E given the occurrence of event F
$\mathbb{E}(X)$	Expectation of a random variable X
$\text{Var}(X)$	Variance of a random variable X
$\text{Cov}(X, Y)$	Covariance between the random variables X and Y
$\text{Bin}(n, p)$	Binomial distribution with parameters n and p; $n \in \mathbb{N}, 0 < p < 1$
$\text{Poisson}(\lambda)$	Poisson distribution with mean λ; $\lambda > 0$
$\text{U}(a, b)$	Continuous uniform distribution on the interval (a, b); $a < b, a, b \in \mathbb{R}$
$\text{Exp}(\lambda)$	Exponential distribution with mean $1/\lambda$; $\lambda > 0$
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2; $\mu \in \mathbb{R}, \sigma > 0$
$N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$	Bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2 and correlation ρ; $\mu_1 \in \mathbb{R}, \mu_2 \in \mathbb{R}, \sigma_1 > 0, \sigma_2 > 0, -1 < \rho < 1$
$\Phi(\cdot)$	The cumulative distribution function of the $N(0,1)$ random variable
χ_n^2	Central chi-square distribution with n degrees of freedom
t_n	Central Student's t distribution with n degrees of freedom
$F_{m,n}$	Central F distribution with m and n degrees of freedom
$\chi_{n,\alpha}^2$	A constant such that $\mathbb{P}(X > \chi_{n,\alpha}^2) = \alpha$, where $X \sim \chi_n^2$, $\alpha \in (0,1)$
$t_{n,\alpha}$	A constant such that $\mathbb{P}(X > t_{n,\alpha}) = \alpha$, where $X \sim t_n$, $\alpha \in (0,1)$
$\xrightarrow{d}$	Convergence in distribution
$\xrightarrow{P}$	Convergence in probability
i.i.d.	Independent and identically distributed
Stirling's approximation	$\Gamma(x+1) \approx \left(\frac{x}{e}\right)^x \sqrt{2\pi x}$ as $x \to \infty$, where Γ is the Gamma function
$[x]$	Greatest integer smaller than or equal to x
$\dim(V)$	Dimension of the vector space V
$\text{rank}(A)$	Rank of the matrix A
A^T	Transpose of the matrix A
f'	Derivative of the function f

Section A: Q.1 – Q.10 Carry ONE mark each.

Q.1 Let $\{a_n\}_{n\geq1}$ and $\{b_n\}_{n\geq1}$ be sequences given by

$$a_n = \left[\frac{n^2}{n+1}\right] \text{ and } b_n = \frac{n^2}{n+1} - a_n .$$

Then

MCQ1

(A) $\{a_n\}_{n\geq1}$ converges and $\{b_n\}_{n\geq1}$ diverges

(B) $\{a_n\}_{n\geq1}$ diverges and $\{b_n\}_{n\geq1}$ converges

(C) Both $\{a_n\}_{n\geq1}$ and $\{b_n\}_{n\geq1}$ diverge

(D) Both $\{a_n\}_{n\geq1}$ and $\{b_n\}_{n\geq1}$ converge

Q.2 Let a, b, c be real numbers with $b \neq c$. Define the matrix

$$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

Then the number of characteristic roots of M that are real is

MCQ1

(A) 3

(B) 2

(C) 1

(D) 0

Q.3 Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous odd function that is not identically zero.

Further, suppose that f is a periodic function. Define

$$g(x)=\int_0^x f(t)\,dt\,.$$

Then

MCQ1

(A) g is odd and not periodic

(B) g is odd and periodic

(C) g is even and not periodic

(D) g is even and periodic

Q.4 Suppose $Z_1, Z_2, \ldots Z_{128}$ are i.i.d. $\text{Bin}(1, 0.5)$ random variables. Define

$$\mathbf{X} = (Z_1, Z_2, \ldots, Z_{64})^T \text{ and } \mathbf{Y} = (Z_{65}, Z_{66}, \ldots, Z_{128})^T.$$

Then the value of $\text{Var}(\mathbf{X}^T\mathbf{Y})$ is

MCQ1

(A) 4

(B) 8

(C) 12

(D) 16

Q.5 Let X_1, X_2, X_3 be i.i.d. $\text{Bin}(1,\theta)$ random variables. Consider the problem of testing the null hypothesis $H_0 : \theta = \frac{1}{2}$ against the alternative hypothesis $H_1 : \theta = \frac{1}{4}$ based on X_1, X_2, X_3. Then the power of the most powerful test of size 0.125 is

MCQ1

(A) 0

(B) $\frac{1}{64}$

(C) $\frac{27}{64}$

(D) $\frac{7}{8}$

Q.6 Suppose X is a Poisson(λ) random variable. Define $Y = (-1)^X X$. Then the expected value of Y is

MCQ1

(A) $-\lambda e^{-2\lambda}$

(B) $-\lambda e^{-\lambda}$

(C) $\lambda e^{-2\lambda}$

(D) $-\lambda$

Q.7 Let $\{Y_k\}_{k\geq 1}$ be a sequence of i.i.d. $\text{Bin}(1, p)$ random variables, where $0 < p < 1$ is an unknown parameter. Let $\hat{p}_n$ be the maximum likelihood estimator of p based on $Y_1, Y_2, \ldots, Y_n$. It is claimed that:

(I) $$\frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} N(0,1) \text{ as } n \to \infty$$

(II) $$\frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} \xrightarrow{d} N(0,1) \text{ as } n \to \infty$$

Which of the following statements is correct?

MCQ1

(A) (I) is correct and (II) is incorrect

(B) (I) is incorrect and (II) is correct

(C) Both (I) and (II) are correct

(D) Both (I) and (II) are incorrect

Q.8 Let X be a continuous random variable with probability density function $f(x)$. Consider the problem of testing the null hypothesis

$$H_0 : f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

against the alternative hypothesis

$$H_1 : f(x) = \begin{cases} 2x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the power of the most powerful size α test, where $0 < \alpha < 1$, based on a single sample, is

MCQ1

(A) $\alpha(1-\alpha)$

(B) $\alpha(2-\alpha)$

(C) $1-\alpha$

(D) α

Q.9 Suppose $X \sim N(0,4)$ and $Y \sim N(0,9)$ are independent random variables. Then the value of $\mathbb{P}(9X^2 + 4Y^2 < 6)$ is

MCQ1

(A) $1-e^{-1/4}$

(B) $1-e^{-1/12}$

(C) $1-e^{-1/6}$

(D) $1-e^{-1/9}$

Q.10 Let X be a single sample from a continuous distribution with probability density function

$$f(x;\theta)=\begin{cases}\dfrac{2(\theta-x)}{\theta^2} & \text{if } 0<x<\theta,\\ 0 & \text{otherwise,}\end{cases}$$

where $\theta>0$ is an unknown parameter. For $0<\alpha<0.05$, a $100(1-\alpha)\%$ confidence interval for θ based on X is

MCQ1

(A) $\left(\dfrac{X}{1-\sqrt{\dfrac{\alpha}{2}}}, \dfrac{X}{1-\sqrt{1-\dfrac{\alpha}{2}}}\right)$

(B) $\left(\dfrac{X}{1-\sqrt{\alpha}}, \dfrac{X}{1-\sqrt{1-\alpha}}\right)$

(C) $\left(\left(1-\sqrt{1-\dfrac{\alpha}{2}}\right)X, \left(1-\sqrt{\dfrac{\alpha}{2}}\right)X\right)$

(D) $\left(\dfrac{\alpha}{2}X, \left(1-\dfrac{\alpha}{2}\right)X\right)$

Section A: Q.11 – Q.30 Carry TWO marks each.

Q.11 Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be given by $f(x)=x+\pi\cos x$. Then the number of solutions of the equation $f(x)=0$ is

MCQ2

(A) 1

(B) 2

(C) 3

(D) 4

Q.12 A fair die is thrown three times independently. The probability that 4 is the maximum value that appears among these throws is equal to

MCQ2

(A) $\frac{8}{27}$

(B) $\frac{1}{216}$

(C) $\frac{37}{216}$

(D) $\frac{1}{2}$

Q.13 Let A be an $n \times n$ matrix. Which of the following statements is NOT necessarily true?

MCQ2

(A) If $\text{rank}(A^5) = \text{rank}(A^6)$, then $\text{rank}(A^6) = \text{rank}(A^7)$

(B) If $\text{rank}(A) = n$, then it is possible to obtain a singular matrix by suitably changing a single entry of A

(C) If $\text{rank}(A) = n$, then $\text{rank}(A + A^T) \geq \dfrac{n}{2}$

(D) If $\text{rank}(A) < n$, then it is possible to obtain a nonsingular matrix by suitably changing $n - \text{rank}(A)$ entries of A

Q.14 Let V be a subspace of $\mathbb{R}^{10}$. Suppose A is a 10×10 matrix with real entries. Let $A^k(V)=\{A^k\mathbf{x}:\mathbf{x}\in V\}$ for $k\geq 1$ and $A(V)=A^1(V)$. Which one of the following statements is NOT true?

MCQ2

(A) If A is nonsingular, then $\dim(V)=\dim(A(V))$ necessarily holds

(B) It is possible that A is singular and $\dim(V)=\dim(A(V))$

(C) If $\text{rank}(A)=8$, then $\dim(A(V))\geq\dim(V)-2$ necessarily holds

(D) If $\dim(V)=\dim(A(V))=\dim(A^2(V))=\cdots=\dim(A^5(V))$, then $\dim(A^6(V))=\dim(V)$ necessarily holds

Q.15 A function $f:(0,1)\to\mathbb{R}$ is said to have property $\mathcal{I}$ if, for any $0<x_1<x_2<1$ and for any c between $f(x_1)$ and $f(x_2)$, there exists $y\in[x_1,x_2]$ such that $f(y)=c$. Consider the following statements:

(I) If $g:(0,1)\to\mathbb{R}$ satisfies property $\mathcal{I}$, then g is necessarily continuous.

(II) If $h:(0,1)\to\mathbb{R}$ is differentiable, then h' necessarily satisfies property $\mathcal{I}$.

Then

MCQ2

(A) (I) is correct and (II) is incorrect

(B) (I) is incorrect and (II) is correct

(C) Both (I) and (II) are correct

(D) Both (I) and (II) are incorrect

Q.16 Suppose f is a polynomial of degree n with real coefficients, and A is an $n \times n$ matrix with real entries satisfying $f(A) = 0$. Consider the following statements:

(I) If $f(0) \neq 0$, then A is necessarily nonsingular.

(II) If $f(0) = 0$, then A is necessarily singular.

Then

MCQ2

(A) (I) is correct and (II) is incorrect

(B) (I) is incorrect and (II) is correct

(C) Both (I) and (II) are correct

(D) Both (I) and (II) are incorrect

Q.17 Let X_1 and X_2 be i.i.d. $N(0, \sigma^2)$ random variables. Define $Z_1 = X_1 + X_2$ and $Z_2 = X_1 - X_2$. Then which one of the following statements is NOT correct?

MCQ2

(A) Z_1 and Z_2 are independently distributed

(B) Z_1 and Z_2 are identically distributed

(C) $\mathbb{P}\left(\frac{Z_1}{Z_2} < 1\right) = 0.5$

(D) $\frac{Z_1}{Z_2}$ and $Z_1^2 + Z_2^2$ are independently distributed

Q.18 Consider a circle C with unit radius and center at $A=(0,0)$. Let $B=(1,0)$. Suppose $\Theta \sim \mathrm{U}(0,\pi)$ and $D=(\cos\Theta, \sin\Theta)$. Note that the angle $\angle DAB=\Theta$. Then the expected area of the triangle ABD is

MCQ2

(A) $\frac{1}{\pi}$

(B) $\frac{2}{\pi}$

(C) $\frac{1}{2\pi}$

(D) 1

Q.19 Suppose $Y \sim \mathrm{U}(0,1)$ and the conditional distribution of X given $Y = y$ is $\mathrm{Bin}(6, y)$, for $0 < y < 1$. Then the probability that $(X+1)$ is an even number is

MCQ2

(A) $\frac{3}{7}$

(B) $\frac{1}{2}$

(C) $\frac{4}{7}$

(D) $\frac{5}{14}$

Q.20 Let X be an $\text{Exp}(\lambda)$ random variable. Suppose $Y = \min\{X, 2\}$. Let F_X and F_Y denote the distribution functions of X and Y respectively. Then which of the following statements is true?

MCQ2

(A) F_Y is a continuous function

(B) $F_Y(y)$ is discontinuous at $y = 2$

(C) $F_Y(t) \le F_X(t)$ for all $t \in \mathbb{R}$

(D) $\mathbb{E}(Y) > \mathbb{E}(X)$

Q.21 Let $X_1, X_2, \ldots, X_5$ be i.i.d. $N(0,\sigma^2)$ random variables. Suppose c is such that

$$\mathbb{E}\left(c\sqrt{\sum_{i=1}^{5} X_i^2} \right) = \sigma.$$

Then the value of c is

MCQ2

(A) $\sqrt{\frac{9\pi}{128}}$

(B) $\sqrt{\frac{9\pi}{64}}$

(C) $\sqrt{\frac{9}{128\pi}}$

(D) $\sqrt{\frac{3}{64\pi}}$

Q.22 Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution with probability density function

$$f(x;\theta) = \frac{1}{2\theta} e^{-\frac{|x|}{\theta}}, \quad x \in \mathbb{R},$$

where $\theta > 0$ is an unknown parameter. The critical region for the uniformly most powerful test for testing the null hypothesis $H_0 : \theta = 2$ against the alternative hypothesis $H_1 : \theta > 2$ at level α, where $0 < \alpha < 1$, is

MCQ2

(A) $$\left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : 2\sum_{i=1}^{n} |x_i| < \chi^2_{2n,1-\alpha} \right\}$$

(B) $$\left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : 2\sum_{i=1}^{n} |x_i| > \chi^2_{2n,\alpha} \right\}$$

(C) $$\left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| < \chi^2_{2n,1-\alpha} \right\}$$

(D) $$\left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| > \chi^2_{2n,\alpha} \right\}$$

Q.23 Let (X,Y) have the $N_2(0,0,1,1,0.25)$ distribution. Then the correlation coefficient between e^X and e^{2Y} is

MCQ2

(A) $\dfrac{e^3 - e^{5/2}}{\left(e^5(e-1)(e^4-1)\right)^{1/2}}$

(B) $\dfrac{e^3 - e^{5/2}}{\left(e^4(e-1)(e^5-1)\right)^{1/2}}$

(C) $\dfrac{e^2 - e^{5/2}}{\left(e^5(e-1)(e^4-1)\right)^{1/2}}$

(D) $\dfrac{e^{3/2} - e^{5/2}}{\left(e^5(e^2-1)(e^4-1)\right)^{1/2}}$

Q.24 Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. $\mathrm{U}(-1,1)$ random variables. Suppose

$$Y_n = \sqrt{3n}\,\frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i^4}.$$

Then $\{Y_n\}_{n\geq 1}$ converges in distribution as $n \to \infty$ to a

MCQ2

(A) $N(0,1)$ random variable

(B) random variable degenerate at 0

(C) $N(0,25)$ random variable

(D) $N(0,0.04)$ random variable

Q.25 If $(X,Y) \sim N_2(0,0,1,1,0.5)$, then the value of $\mathbb{E}\left(e^{-XY}\right)$ is

MCQ2

(A) $\frac{2}{\sqrt{5}}$

(B) $\frac{2}{\sqrt{3}}$

(C) $\frac{1}{\sqrt{2}}$

(D) $\frac{1}{2}$

Q.26 Let $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$ be a random sample from a $N_2(0,0,1,1,\rho)$ distribution, where ρ is an unknown parameter. Which of the following statements is NOT correct?

MCQ2

(A) $\left(\sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_i Y_i\right)$ is a sufficient statistic for ρ

(B) $\left(\sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_i Y_i\right)$ is not a minimal sufficient statistic for ρ

(C) $\sum_{i=1}^{n} X_i^2$ is an ancillary statistic

(D) $\left(\sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} Y_i^2\right)$ is an ancillary statistic

Q.27 Let $(Y_1, Y_2, Y_3) \in \{0, 1, \ldots, n\}^3$ be a discrete random vector having joint probability mass function

$$\mathbb{P}(Y_1 = y_1, Y_2 = y_2, Y_3 = y_3) = \begin{cases} \dfrac{n!}{y_1!\, y_2!\, y_3!} p^{y_1} (2p)^{y_2} (1-3p)^{y_3} & \text{if } y_1 + y_2 + y_3 = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \le p \le \frac{1}{3}$ is an unknown parameter. Assume the convention $0^0 = 1$. The maximum likelihood estimator of p is denoted by $\hat{p}$. Which of the following statements is correct?

MCQ2

(A) $\mathbb{E}(\hat{p}) > p$

(B) $\hat{p}$ is an unbiased estimator of p, but not the uniformly minimum variance unbiased estimator of p

(C) $\hat{p}$ is the uniformly minimum variance unbiased estimator of p

(D) $\mathbb{E}(\hat{p}) < p$

Q.28 Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(0,1)$ random variables, where $n > 3$. If

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2,$$

then $\mathrm{Var}\left(\frac{\bar{X}}{S}\right)$ is equal to

MCQ2

(A) $\frac{(n-3)}{n(n-1)}$

(B) $\frac{(n-1)}{n(n-3)}$

(C) $\frac{(n-1)}{n(n-2)}$

(D) $\frac{(n-2)}{n(n-1)}$

Q.29 Suppose (X_i, Y_i), $i = 1, 2, \ldots, 200$, are i.i.d. random vectors each having joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{25\pi} & \text{if } x^2 + y^2 \le 25, \\ 0 & \text{otherwise.} \end{cases}$$

Let M be the cardinality of the set $\{i \in \{1, 2, \ldots, 200\} : X_i^2 + Y_i^2 \le 0.25\}$. Then $\mathbb{P}(M \ge 1)$ is closest to

MCQ2

(A) $\Phi(0.5)$

(B) $1 - e^{-1}$

(C) $\Phi(3)$

(D) $1 - e^{-2}$

Q.30 Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables, where $X_n \sim \text{Bin}(n, p_n)$ with $p_n \in (0,1)$. Which of the following conditions implies that $X_n \xrightarrow{d} 0$ as $n \to \infty$?

MCQ2

(A) $\lim_{n\to\infty} p_n = 0$

(B) $\lim_{n\to\infty} \mathbb{P}(X_n = k) = 0$ for each $k \in \mathbb{N}$

(C) $\lim_{n\to\infty} \mathbb{E}(X_n) = 0$

(D) $\sup_{n\geq 1} \text{Var}(X_n) < \infty$

Section B: Q.31 – Q.40 Carry TWO marks each.

Q.31 Let $\{x_n\}_{n\geq 1}$ be a sequence given by

$$x_n = \frac{2}{3}\left(x_{n-1} + \frac{2}{x_{n-1}}\right), \quad \text{for } n \geq 2,$$

with $x_1 = -10$. Then which of the following statement(s) is/are correct?

MSQ2

(A) $\{x_n\}_{n\geq 1}$ converges

(B) $\{x_n\}_{n\geq 1}$ diverges

(C) $x_{2025} - x_{2024}$ is positive

(D) $x_{2025} - x_{2024}$ is negative

Q.32 Let $a, b \in \mathbb{R}$. Consider the system of linear equations

$$x + y + 3z = 5,$$
$$a\,x - y + 4z = 11,$$
$$2x + b\,y + z = 3.$$

Then which of the following statements is/are correct?

MSQ2

(A) There are finitely many pairs (a,b) such that the system has a unique solution

(B) There are finitely many pairs (a,b) such that the system has no solution

(C) There are finitely many pairs (a,b) such that the system has infinitely many solutions

(D) If $a = b = 1$, the system has no solution

Q.33 Suppose $f:(0,\infty)\to(0,\infty)$ is continuously differentiable. Assume further that $\lim_{x\to\infty} f(x)=0$. Which of the following statements is/are necessarily true?

MSQ2

(A) $\lim_{x\to\infty} f'(x)$ exists and is equal to 0

(B) $\limsup_{x\to\infty} f'(x)=0$

(C) $\liminf_{x\to\infty} f'(x)=0$

(D) $\liminf_{x\to\infty} |f'(x)|=0$

Q.34 The joint moment generating function of (X,Y) is given by

$$M_{X,Y}(s,t)=\left(\frac{1}{4}+\frac{1}{2}e^{s}+\frac{1}{4}e^{t}\right)^{2}, \quad (s,t)\in\mathbb{R}^{2}.$$

Then which of the following statements is/are correct?

MSQ2

(A) $\mathbb{E}(X)=1$

(B) $\mathbb{E}(Y^{2})=\frac{3}{8}$

(C) $\mathrm{Cov}(X,Y)=-\frac{1}{4}$

(D) $\mathrm{Var}(X)=\frac{1}{2}$

Q.35 The joint probability density function of X and Y is given by

$$f(x,y)=\begin{cases} c(x+y) & \text{if } 0\le x,\, y\le 1, \\ 0 & \text{otherwise,} \end{cases}$$

for some constant c. Which of the following statements is/are correct?

MSQ2

(A) $c=1$

(B) X and Y are independent

(C) The probability density function of X is $g(x)=\begin{cases} 2x & \text{if } 0\le x\le 1, \\ 0 & \text{otherwise.} \end{cases}$

(D) $X+Y$ has a probability density function

Q.36 Let $X_1, X_2, \ldots, X_{20}$ be a random sample from a $N(\mu, \sigma^2)$ population. Suppose

$$P = \frac{1}{10}\sum_{i=1}^{10} X_i \text{ and } Q = \frac{1}{9}\sum_{i=1}^{10}(X_i - P)^2 .$$

Then which of the following statements is/are correct?

MSQ2

(A) $\dfrac{X_{11} + P - X_{12} - X_{20}}{\sqrt{Q}} \sim \sqrt{\dfrac{31}{10}}\, t_9$

(B) $\dfrac{P - X_{15}}{\sqrt{9Q + (X_{18} - \mu)^2}} \sim \dfrac{\sqrt{11}}{10}\, t_{10}$

(C) $\dfrac{(X_{12} - X_{20})^2}{Q} \sim 2\, F_{1,9}$

(D) $\dfrac{P - X_{14}}{\sqrt{Q}} \sim \dfrac{\sqrt{11}}{10}\, t_9$

Q.37 Let $X_1, X_2, \ldots, X_n$, where $n > 1$, be a random sample from a $N(\theta, \theta)$ distribution, where $\theta > 0$ is an unknown parameter. Suppose

$$T_n = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } S_n^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - T_n\right)^2 .$$

Then which of the following statements is/are correct?

MSQ2

(A) $T_n\, S_n^2$ is a consistent estimator for θ^2

(B) $T_n^4\, S_n^{-4}$ is a consistent estimator for θ^2

(C) $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ is a complete statistic

(D) $\sum_{i=1}^{n} X_i^2$ is a complete sufficient statistic for θ

Q.38 Let $X_1, X_2, \ldots, X_n$, where $n > 1$, be a random sample from a continuous distribution with probability density function

$$f(x;\theta) = \begin{cases} \theta\, x^{\theta-1} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Then which of the following statistics is/are sufficient for θ?

MSQ2

(A) $\left(X_1, X_2, \ldots, X_n\right)$

(B) $\left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right)$, where $X_{(r)}$ is the r^{th} order statistic, $r = 1, \ldots, n$

(C) $\sum_{i=1}^{n} X_i$

(D) $\prod_{i=1}^{n} X_i$

Q.39 A simple linear regression model $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, with $x_i = (-1)^i$ for $i = 1, 2, \ldots, 20$, is fitted. The random error variables ϵ_i are uncorrelated with mean 0 and finite variance $\sigma^2 > 0$. Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the least squares estimators of β_0 and β_1 respectively. Let $\hat{Y}_i$ be the fitted value of the i^{th} response variable Y_i, for $i = 1, \ldots, 20$. Which of the following statements is/are correct?

MSQ2

(A) $\text{Cov}\left(\hat{\beta}_0, \hat{\beta}_1\right) = 0$

(B) $\text{Var}\left(\hat{\beta}_0\right) = \text{Var}\left(\hat{\beta}_1\right)$

(C) $\text{Var}\left(\hat{\beta}_0\right) = \text{Cov}\left(\hat{Y}_i, \hat{\beta}_0\right)$ for all $i = 1, \ldots, 20$

(D) $\text{Var}\left(\hat{\beta}_1\right) = \text{Cov}\left(\hat{Y}_i, \hat{\beta}_1\right)$ for all $i = 1, \ldots, 20$

Q.40 Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. discrete random variables from a population with probability mass function

$$\mathbb{P}(Y = y; \theta) = \begin{cases} \theta(1-\theta)^y & \text{if } y \in \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \theta \le 1$ is an unknown parameter. Assume the convention $0^0 = 1$. If $\hat{\theta}$ is the method of moments estimator of θ, then which of the following statements is/are correct?

MSQ2

(A) $\hat{\theta}$ is also the maximum likelihood estimator of θ

(B) $\hat{\theta}$ is an unbiased estimator of θ

(C) $\hat{\theta}$ is a consistent estimator of θ

(D) $\dfrac{1}{\hat{\theta}}$ is an unbiased estimator of $\dfrac{1}{\theta}$

Section C: Q.41 – Q.50 Carry ONE mark each.

Q.41 Let A be a 7×7 real matrix with $\text{rank}(A) = 1$. Suppose the trace of A^2 is 2025. Let the characteristic polynomial of A be written as

$$\sum_{n=0}^{7} a_n x^n.$$

Then $\sum_{n=0}^{7} |a_n|$ is ______

(answer in integer)

NAT1

Q.42 The radius of convergence of the power series

$$\sum_{n\geq 1} \frac{n!}{n^n} x^n$$

is equal to _____

(round off to 2 decimal places)

NAT1

Q.43 Let

$$f(x) = x\sin\left(\frac{\pi}{2x}\right), \ x > 0.$$

Then

$$\lim_{h\to 0} \frac{1}{\pi^2 h^2}\left[3f(1) - 2f(1+h) - f(1-2h)\right]$$

is equal to ________________

(round off to 2 decimal places)

NAT1

Q.44 Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1)=\mu$ and $\text{Var}(X_1)=\sigma^2<\infty$. Suppose c is a constant that does not depend on n such that

$$\frac{c}{n}\sum_{i=1}^{n}(X_{2i}-X_{2i-1})^2$$

is a consistent estimator of σ^2. Then c is equal to ________________________ (round off to 2 decimal places)

NAT1

Q.45 Let $X_1, X_2, \ldots, X_{10}$ be i.i.d. $U(0, \theta)$ random variables, where $\theta > 0$ is unknown. For testing the null hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta = 0.9$, consider a test that rejects H_0 if

$$X_{(10)} = \max\{X_1, X_2, \ldots, X_{10}\} < 0.8.$$

Then the probability of type I error of the test is equal to _____

(round off to 2 decimal places)

NAT1

Q.46 Let $(X_1,Y_1),(X_2,Y_2),\ldots,(X_{100},Y_{100})$ be i.i.d. discrete random vectors each having joint probability mass function

$$\mathbb{P}(X=x,Y=y)=\frac{e^{-(1+x)\lambda}((1+x)\lambda)^y}{y!}p^x(1-p)^{1-x}, \quad x\in\{0,1\}, \quad y\in\mathbb{N}\cup\{0\},$$

where $\lambda>0$ and $0<p<1$ are unknown parameters. If the observed values of $\sum_{i=1}^{100}X_i$ and $\sum_{i=1}^{100}Y_i$ are 54 and 521 respectively, the maximum likelihood estimate of λ is equal to _____

(round off to 2 decimal places)

NAT1

Q.47 Let X and Y be i.i.d. random variables with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $Z = \min\{X, Y\}$, then $\mathbb{E}(Z)$ is equal to ____________

(answer in integer)

NAT1

Q.48 The joint probability density function of the random vector (X, Y, Z) is given by

$$f(x,y,z) = \begin{cases} \frac{1}{x\,y} & \text{if } 0 < z < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the value of $\mathbb{P}(X > 5Y)$ is equal to ____________

(round off to 2 decimal places)

NAT1

Q.49 Suppose U_1 and U_2 are i.i.d. $\mathrm{U}(0,1)$ random variables. Further, let X be a $\mathrm{Bin}(2, 0.5)$ random variable that is independent of (U_1, U_2). Then

$$36\,\mathbb{P}(U_1 + U_2 > X)$$

is equal to ______________

(answer in integer)

NAT1

Q.50 A drawer contains 5 pairs of shoes of different sizes. Assume that all 10 shoes are distinguishable. A person selects 5 shoes from the drawer at random. Then the probability that there are exactly 2 complete pairs of shoes among these 5 shoes is equal to ___________________

(round off to 2 decimal places)

NAT1

Section C: Q.51 – Q.60 Carry TWO marks each.

Q.51 Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 12\}$ be a circle in the plane. Let (a,b) be the point on C which minimizes the distance to the point $(1,2)$. Then $b-a$ is ____ (round off to 2 decimal places)

NAT2

Q.52 Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^3+2x^2-15x$ and $g(x)=x$ respectively. Let x_0 be the smallest strictly positive number such that $f(x_0)=0$. Then the area of the region enclosed by the graphs of f and g between the lines $x=0$ and $x=x_0$ is _________

(round off to 2 decimal places)

NAT2

Q.53 Let V be the volume of the region

$$\left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + \frac{z^2}{4} \le 1 \text{ and } |z| \le 1\right\}.$$

Then $\dfrac{V}{\pi}$ is equal to ______

(round off to 2 decimal places)

NAT2

Q.54 Suppose $X_1, X_2, \ldots, X_{10}, Y_1, Y_2, \ldots, Y_{10}$ are independent random variables, where $X_i \sim N(0, \sigma^2)$ and $Y_i \sim N(0, 3\sigma^2)$ for $i = 1, 2, \ldots, 10$. The observables are $D_1, \ldots, D_{10}$, where D_i denotes the Euclidean distance between the points $(X_i, Y_i, 0)$ and $(0, 0, 5)$ for $i = 1, 2, \ldots, 10$. If the observed value of $\sum_{i=1}^{10} D_i^2$ is equal to 1050, then the method of moments estimate of σ^2 is equal to _______ (answer in integer)

NAT2

Q.55 Consider a sequence of independent Bernoulli trials with success probability $p = \frac{1}{7}$. Then the expected number of trials required to get two consecutive successes for the first time is equal to ____________

(answer in integer)

NAT2

Q.56 Let X be a real valued random variable with $\mathbb{E}(X)=1$, $\mathbb{E}(X^2)=4$, $\mathbb{E}(X^4)=16$. Then $\mathbb{E}(X^3)$ is equal to ______________

(answer in integer)

NAT2

Q.57 Let X_1, X_2, X_3, X_4 be a random sample from a continuous distribution with probability density function

$$f(x;\theta) = \begin{cases} 2\,\theta^2\, x^{-3} & \text{if } \theta < x < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. It is known that $X_{(1)} = \min\{X_1, X_2, X_3, X_4\}$ is a complete sufficient statistic for θ. If the observed values are $x_1 = 15, x_2 = 11, x_3 = 10, x_4 = 17$, the uniformly minimum variance unbiased estimate of θ^2 is equal to ________

(answer in integer)

NAT2

Q.58 Let X be a random variable with probability density function

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let δ denote the conditional expectation of X given that $X \leq \frac{1}{2}$.

Then the value of 80δ is equal to __________

(answer in integer)

NAT2

Q.59 Let $X_1, X_2, \ldots, X_7$ be i.i.d. continuous random variables with median θ. If $X_{(1)} < X_{(2)} < \cdots < X_{(7)}$ are the corresponding order statistics, then $\mathbb{P}\left(X_{(2)} > \theta\right)$ is equal to ____________________

(round off to 3 decimal places).

NAT2

Q.60 Suppose (X, Y) has the $N_2(3, 0, 4, 1, 0.5)$ distribution. Then

$4\,\mathrm{Cov}(X+Y, Y^3)$ is equal to ________________

(answer in integer)

NAT2