MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let $P_n = \alpha^n + \beta^n$, $P_{10} = 123$, $P_9 = 76$, $P_8 = 47$ and $P_1 = 1$, then quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is

(1) $x^2 + x - 1 = 0$ (2) $x^2 - 2x + 1 = 0$

(3) $x^2 + x - 2 = 0$ (4) $x^2 - x - 2 = 0$

Answer (1)

Sol. $\because P_{10} = P_9 + P_8$

$\Rightarrow P_{10} - P_9 - P_8 = 0$

By Newton's method

Therefore, the equation is

$x^2 - x - 1 = 0$

as $P_1 = 1$

$\Rightarrow \alpha + \beta = 1$ and $\alpha\beta = 1$

$\therefore$ Quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{\alpha+\beta}{\alpha\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - (-1)x - 1 = 0$$

$$\Rightarrow x^2 + x - 1 = 0$$

2. Let $a_1, a_2, a_3, \ldots$ is in A.P. and $\sum_{k=1}^{12} a_{2k-1} = -\frac{72}{5}a_1$ and $\sum_{k=1}^{n} a_k = 0$. Then the value of n is

(1) 8 (2) 10

(3) 11 (4) 13

Answer (3)

Sol. $\sum_{k=1}^{12} a_{2k-1} = -\frac{72}{5}a_1$

$$a_1 + a_3 + \cdots + a_{23} = -\frac{72}{5}a_1$$

$$a + a + 2d + \cdots + a + 22d = -\frac{72}{5}a$$

$$12a + 2d(1 + 2 + \cdots 11) = -\frac{72}{5}a$$

$$\Rightarrow 12a + 2d\left(\frac{11\times 12}{2}\right) = -\frac{72}{5}a$$

$$\Rightarrow 132d = -\frac{132}{5}a$$

$\Rightarrow a = -5d$ ...(i)

Also $\sum_{k=1}^{n} a_k = 0$

$\Rightarrow S_n = 0$

$\Rightarrow \frac{n}{2}[2a + (n-1)d] = 0$

$\Rightarrow 2a = -(n-1)d$ ...(ii)

From equation (i) and (ii)

$(n-1)d = 10d$

$\therefore n = 11$

3.

4. Given the equation of a hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its directrix is $x = \sqrt{\frac{10}{81}}$ with a focus at $\left(\sqrt{10}, 0\right)$, then find the value of $9(e + l^2)$, where l is length of latus rectum is

(1) 2697 (2) 2597

(3) 2487 (4) 2587

Answer (4)

Sol. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Directrix : $x=\frac{\sqrt{10}}{9}=\frac{a}{e}$ ...(i)

Focus : $(ae,0)\Rightarrow ae=\sqrt{10}$ ...(ii)

(i) × (ii)

$a^2=\frac{10}{9}$

Dividing equation (ii) by (i) we get,

$e^2=9\Rightarrow e=3$

Also, $e^2=9=1+\frac{9b^2}{10}$

$\Rightarrow\ b^2=\frac{80}{9}$

$l=2\times\frac{b^2}{a}=\frac{2\times\frac{80}{9}}{\frac{\sqrt{10}}{3}}=\frac{160}{3\sqrt{10}}$

Now, $9\left(e+l^2\right)=9\left[3+\frac{25600}{9\times10}\right]$

$9\left(3+\frac{160}{3\sqrt{10}}\right)=27+2560$

$=2587$

5. If a twice differentiable function f satisfies $f'(x)=f(x)$ such that $f(0)=\frac{1}{2}=f'(0)$. Then $f''\left(\frac{\pi}{3}\right)$ equals to

(1) $e^{\frac{\pi}{3}}$ (2) $\frac{e^{\frac{\pi}{3}}}{2}$

(3) $\frac{\sqrt{3}}{2}$ (4) $\frac{e^{\frac{2\pi}{3}}}{2}$

Answer (2)

Sol. If $f'(x)=f(x)$

Let $f(x)=y$

$\Rightarrow\ f'(x)dx=dy$

$\frac{dy}{dx}=y\ \Rightarrow \ln y=\int dx$

$\Rightarrow \ln y=x+c$

$\Rightarrow y=e^{x+c}=ke^x$, where $k=e^c$

Since $f(0)=\frac{1}{2}$

$\Rightarrow \frac{1}{2}=ke^0\Rightarrow k=\frac{1}{2}$

$\Rightarrow y=f(x)=\frac{e^x}{2}$

Now, $f'(x)=\frac{e^x}{2}\Rightarrow f'(0)=\frac{1}{2}$ verified

Now, $f''(x)=\frac{e^x}{2}$ at $\frac{\pi}{3}$ will be

$f''\left(\frac{\pi}{3}\right)=\frac{e^{\frac{\pi}{3}}}{2}$

6. Let the system of equations, $3x-y+\beta z=3$, $2x+\alpha y+z=-3$ and $x+y+4z=4$ has infinite solutions, then $22\beta-9\alpha$ equals to

(1) 165 (2) 164

(3) 163 (4) 162

Answer (2)

Sol. $3x-y+\beta z=3$

$2x+\alpha y+z=-3$

$x+y=4z=4$

has infinite solution

$\Rightarrow\ \Delta=0=\Delta_1=\Delta_2=\Delta_3$

$\Delta=0\Rightarrow\begin{vmatrix}3&-1&\beta\\2&\alpha&1\\1&1&4\end{vmatrix}=0$

$\Rightarrow\ 2x-xy+12y+4=0$

$\Delta_2=0\Rightarrow\begin{vmatrix}3&\beta&3\\2&1&-3\\1&4&4\end{vmatrix}=0$

$\Rightarrow\ -11\beta+69=0$

$\beta=\frac{69}{11}$

$$\Delta_3 = 0 \Rightarrow \begin{vmatrix} 3 & -1 & 3 \\ 2 & \alpha & -3 \\ 1 & 1 & 4 \end{vmatrix} = 0$$

$9\alpha + 26 = 0$

$\alpha = -\frac{26}{9}$

$22\beta - 9\alpha = 22 \times \frac{69}{11} + \frac{26}{9} \times 9 = 164$

7. If $\lim\limits_{x\to 0} \frac{(\gamma-1)e^{x^2} + x^2\sin(\alpha x)}{\sin(2x) - \beta x} = 3$, then $\alpha + 2\beta + \gamma$ is equal to

(1) 0 (2) 1

(3) 3 (4) 5

Answer (2)

Sol. At $x \to 0$

$\sin 2x - \beta x \to 0$

$\Rightarrow \frac{0}{0}$ form

$\Rightarrow (\gamma - 1)e^0 + 0\sin(\alpha x) \to 0$

$\Rightarrow (\gamma - 1) = 0$

$\Rightarrow \gamma = 1$

$$\Rightarrow \lim_{x\to 0} \frac{x^2 \sin(\alpha x)}{(\sin 2x - \beta x)} = 3$$

$$\Rightarrow \lim_{x\to 0} \frac{x^2\left[\alpha x - \frac{(\alpha x)^3}{3!} + \frac{(\alpha x)^5}{5!} - \cdots\right]}{\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}\right] - \beta x}$$

$$\Rightarrow \lim_{x\to 0} \frac{\alpha x^3 - \frac{\alpha^3 x^5}{3!} + \frac{\alpha^5 x^7}{5!} - \cdots}{x(2-\beta) - \frac{8x^3}{6} + \frac{2^5 \cdot x^5}{5!} - \cdots} = 3$$

$\Rightarrow 2 - \beta = 0$ and $\frac{\alpha}{\frac{-8}{6}} = 3$

$\Rightarrow \beta = 2$

$\alpha = 3\left(-\frac{8}{6}\right) = -4$

$\Rightarrow \gamma = 1, \beta = 2, \alpha = -4$

$\Rightarrow \alpha + 2\beta + \gamma = 1 + 4 - 4 = 1$

8. The term independent of x in the binominal expression of $\left(\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}}\right)^{10}$ is

(1) 120 (2) 210

(3) 84 (4) 110

Answer (2)

Sol. $(x+1) = \left[\left(x^{\frac{1}{3}}\right) + 1\right]\left[x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1\right]$

$(x+1) = \left(\sqrt{x} - 1\right)\left(\sqrt{x} + 1\right)$

Now

$$\left(\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1}\right) = \left(x^{\frac{1}{3}} + 1\right)$$

and

$$\frac{x-1}{x - x^{\frac{1}{2}}} = \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\left(\left(\sqrt{x}\right)^2 - \sqrt{x}\right)}$$

$$= \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\left(\sqrt{x}\right)\left(\sqrt{x}-1\right)} = 1 + \frac{1}{\sqrt{x}}$$

$\Rightarrow$ The expansion become

$$\left[\left(x^{\frac{1}{3}} + 1\right) - \left(1 + \frac{1}{\sqrt{x}}\right)\right]^{10} = \left(x^{\frac{1}{3}} - \frac{1}{x^{\frac{1}{2}}}\right)^{10}$$

The general term of the expansion with be

$$T_{r+1} = {}^{10}C_r\left(-\frac{1}{x^{\frac{1}{2}}}\right)^r \cdot \left(x^{\frac{1}{3}}\right)^{10-r}$$

$$= {}^{10}C_r \frac{(-1)^r}{x^{\frac{r}{2}}} x^{\left(\frac{10-r}{3}\right)}$$

$$\Rightarrow x^{\left(\frac{10-r}{3} - \frac{r}{2}\right)} \cdot {}^{10}C_r(-1)^r$$

The term independent when exponent of x is 0

$\Rightarrow \quad \frac{10-r}{3} = \frac{r}{2} \Rightarrow r = 4$

$\Rightarrow$ The term independent of x will be

${}^{10}C_4\,(-1)^4 x^0 = 210$

9. Let E be an ellipse such that $E : \frac{x^2}{18} + \frac{y^2}{9} = 1$. Let point P lies on E such that S and S' are foci of ellipse. Then the sum of min $(PS \cdot PS')$ + max $(PS \cdot PS')$ is

(1) 18 (2) 36

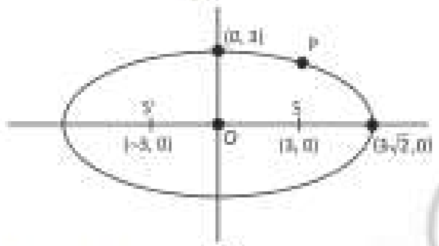
(3) 9 (4) 27

Answer (4)

Sol. For $E = \frac{x^2}{18} + \frac{y^2}{9} = 1$.

$a = 3\sqrt{2}, b = 3 \Rightarrow e = \frac{1}{\sqrt{2}}$

$\Rightarrow$



Since, $PS + PS' = 2a = 6\sqrt{2}$

$\Rightarrow \frac{PS + PS'}{2} \geq \sqrt{PS \cdot PS'}$

$\Rightarrow \left(3\sqrt{2}\right)^2 \geq PS \cdot PS'$

$\Rightarrow (PS \cdot PS')_{max} = 18$

and it happens when P lies on the minor axis similarly minima happen P lies on major axis.

$\Rightarrow P = \left(3\sqrt{2}, 0\right)$

$PS = \left(3\sqrt{2} - 3\right)$

$PS' = \left(3\sqrt{2} + 3\right)$

$\Rightarrow (PS \cdot PS')_{min} = \left(3\sqrt{2}\right)^2 - 3^2$

$= 18 - 9 = 9$

$\Rightarrow (PS \cdot PS')_{min} + (PS \cdot PS')_{max} = 18 + 9 = 27$

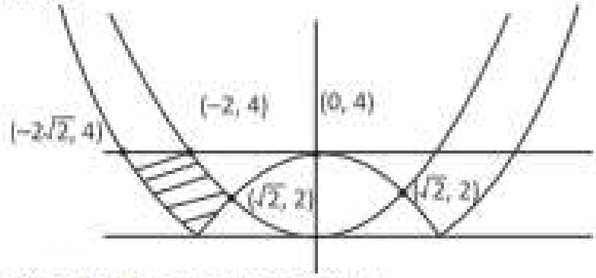
10. The area enclosed by $|4 - x^2| \leq y \leq x^2; y \leq 4, x \leq 0$ equals to (in square units)

(1) $\frac{4}{3}(20\sqrt{2} - 24)$ (2) $\frac{2}{3}(20\sqrt{2} + 24)$

(3) $\frac{4}{3}(10\sqrt{2} + 24)$ (4) $\frac{2}{3}(20\sqrt{2} - 24)$

Answer (4)

Sol.



Shaded region is the required area.

$$\text{Area} = \int_0^2 \left(-\sqrt{4-y} + \sqrt{4+y}\right) dy + \int_2^4 \left(-\sqrt{y} + \sqrt{4+y}\right) dy$$

$$= \left[\frac{2(4+y)^{3/2}}{3} + \frac{2(4-y)^{3/2}}{3}\right]_0^2 + \left[\frac{2(4+y)^{3/2}}{3} - \frac{2y^{3/2}}{3}\right]_0^4$$

$$= \frac{2}{3}\left[(6^{3/2} + 2^{3/2}) - (8 + 8) + (8^{3/2} - 8) - (6^{3/2} - 2^{3/2})\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 16 + 8^{3/2} - 8 + 2^{3/2}\right]$$

$$= \frac{2}{3}\left[4\sqrt{2} + 16\sqrt{2} - 24\right]$$

$$= \frac{2}{3}\left[20\sqrt{2} - 24\right] \text{ sq. unit}$$

11. Let $\theta \in [-2\pi, 2\pi]$ satisfying $2\cos^2\theta - \sin\theta - 1 = 0$. Then the number of solutions of equation is

(1) 2 (2) 4

(3) 6 (4) 8

Answer (3)

Sol. $2\cos^2\theta - \sin\theta - 1 = 0$

$2(1 - \sin^2\theta) - \sin\theta - 1 = 0$

$\Rightarrow \quad -2\sin^2\theta - \sin\theta + 1 = 0$

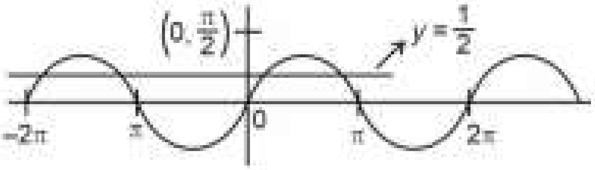
$\Rightarrow \quad 2\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \quad 2\sin^2\theta + 2\sin\theta - \sin\theta - 1 = 0$

$\Rightarrow$ $2\sin\theta\,(\sin\theta + 1) - 1\,(\sin\theta + 1) = 0$

$\Rightarrow$ $(2\sin\theta - 1)\,(\sin\theta + 1) = 0$

$\Rightarrow$ $\sin\theta = \frac{1}{2},\ -1$



Total 6 solutions possible.

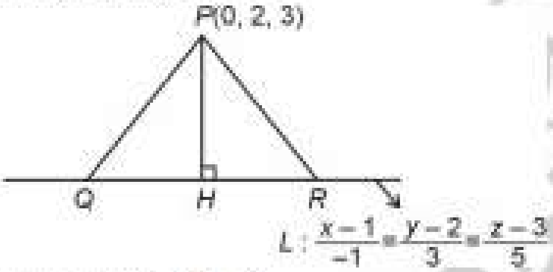
12. If Q and R are two points on line $L : \frac{x-1}{-1} = \frac{y-2}{3} = \frac{z-3}{5}$ such that $QR = 5$. If $P(0, 2, 3)$ be any point, then the area of ΔPQR is

(1) $\sqrt{\frac{85}{14}}$ (2) $\sqrt{\frac{75}{14}}$

(3) $\frac{\sqrt{85}}{14}$ (4) $\frac{\sqrt{75}}{14}$

Answer (1)

Sol. H is point on the line L



$H(-K + 1, 3K + 2, 5K + 3)$

DR's of PH are $-K + 1, 3K, 5K$

$PH \perp L$

$-(-K + 1) + 3\,(3K) + 5K(5) = 0$

$K - 1 + 9K + 25K = 0$

$35K = 1$

$K = \frac{1}{35}$

$H\left(\frac{34}{35}, \frac{73}{35}, \frac{110}{35}\right)$

$PH = \sqrt{\left(\frac{34}{35} - 0\right)^2 + \left(\frac{73}{35} - 2\right)^2 + \left(\frac{110}{35} - 3\right)^2}$

$PH = \sqrt{\frac{34}{35}}$

As $(\Delta PQR) = \frac{1}{2} \times PH \times QR$

$= \frac{1}{2} \times \sqrt{\frac{34}{35}} \times 5$

$= \sqrt{\frac{85}{14}}$ sq. unit

13. Let $\sin x \cos y\,(f(2x + 2y) - f(2x - 2y)) = \cos x \sin y\,(f(2x + 2y) + f(2x - 2y))$ $\forall\, x, y \in R$ and $f'(0) = \frac{1}{2}$. If $f(x)$ is differentiable function, then $f''\left(\frac{2\pi}{3}\right)$ is

(1) $\frac{1}{8}$ (2) $\frac{3}{8}$

(3) $-\frac{1}{16}$ (4) $-\frac{3}{4}$

Answer (3)

Sol. $f(2x + 2y)\,[\sin x \cos y - \cos x \sin y]$

$- f(2x - 2y)\,[\sin x \cos y + \cos x \sin y] = 0$

$\Rightarrow$ $\sin(x - y)\,f(2x + 2y) = f(2x - 2y)\,\sin(x + y)$

$\Rightarrow$ $\frac{f(2x + 2y)}{\sin(x + y)} = \frac{f(2x - 2y)}{\sin(x - y)} = k$ (say)

$\therefore$ $f(2x + 2y) = k\sin(x + y)$

$f(2x) = k\sin x$ $(\because\ y = 0)$

Hence, $f(x) = k\sin\frac{x}{2}$

$f'(x) = \frac{k}{2}\cos\frac{x}{2}$

$f'(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{k}{2}$

$\Rightarrow$ $k = 1$

$\boxed{f(x) = \sin\frac{x}{2}}$

$f'(x) = \frac{1}{2}\cos\frac{x}{2}$

$f''(x) = -\frac{1}{4}\sin\frac{x}{2}$

$f'''(x) = -\frac{1}{8}\cos\frac{x}{2}$

$f'''\left(\frac{2\pi}{3}\right) = -\frac{1}{8}\cos\left(\frac{\pi}{3}\right)$

$= -\frac{1}{8}\times\frac{1}{2} = -\frac{1}{16}$

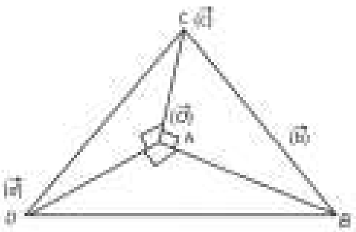
14. For a tetrahedron *ABCD*, the area of triangular face *ABC*, *ACD* and *ABD* is 5, 6 and 7 sq. units respectively. If *AB*, *AC* and *AD* are mutually orthogonal, then the area of triangular face *BCD* is

(1) $\sqrt{11}$ sq. units (2) $\sqrt{110}$ sq. units

(3) $\sqrt{550}$ sq. units (4) $\sqrt{55}$ sq. units.

Answer (2)

Sol.



$\Rightarrow$ Using de gua's theorem

$\Rightarrow [Ar(\Delta BCD)]^2 = Ar(\Delta ABC)^2$

+ Ar(ΔACD)2 + Ar(ΔADB)2

$= 5^2$ + 62 + 72 = 110

$\Rightarrow$ Ar (ΔBCD) = $\sqrt{110}$ sq. units

After:

$\Delta ABC = \frac{1}{2}|\vec{b}\times\vec{c}| = \frac{1}{2}|\vec{b}||\vec{c}|$

$\Delta ACD = \frac{1}{2}|\vec{c}\times\vec{d}| = \frac{1}{2}|\vec{c}||\vec{d}|$

$\Delta ABD = \frac{1}{2}|\vec{b}\times\vec{d}| = \frac{1}{2}|\vec{b}||\vec{d}|$

$\Rightarrow \text{Area}(\Delta ABD) = \frac{1}{2}\left|(\vec{b}-\vec{d})\times(\vec{c}-\vec{d})\right|$

$= \frac{1}{2}\left|\vec{b}\times\vec{c}+\vec{d}\times\vec{b}+\vec{c}\times\vec{d}\right|$

$\Rightarrow Ar(BCD)^2 = Ar(ABC)^2 + Ar(ACD)^2 + Ar(ABD)^2$

15. If $\frac{2+k^2z}{k+k\bar{z}} = z, k \neq 0$, such that $z = x + iy$ and $y \neq 0$ and $|z-1+2i| = 1$, then the maximum distance of point $(k + k^2i)$ from the given circle on which *z* lies is

(1) 2

(2) 4

(3) 3

(4) 1

Answer (2)

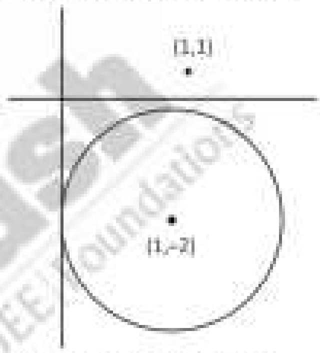
Sol. $2 + k^2z = zk + k|z|^2$

$z(k^2-k) = k|z|^2 - 2$

If *z* is not purely real

$\Rightarrow$ $k^2 - k \Rightarrow k = 0$ or $1, \Rightarrow k = 1$

$\Rightarrow$ Point is $k + k^2i = (1 + i)$



The maximum distance is

$\sqrt{(1-1)^2+(1+2)^2}$ + radius = 3 + 1 = 4

16. Let C_1 and C_2 are circle passing through (–9, 4), both are in contact with $x + y = 3$ and $x - y = 3$ (tangent lines). If r_1 and r_2 are radius of C_1 and C_2 respectively, then $|r_1^2 - r_2^2|$ equals to

(1) 400

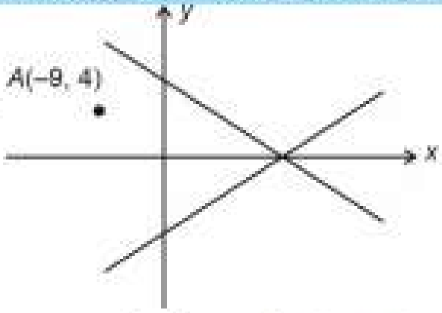
(2) 768

(3) 625

(4) 250

Answer (2)

Sol.



$\because$ $x + y = 3$ and $x - y = 3$ are tangents

$\therefore$ Both circle centre will lie on x-axis

$\therefore$ $(x - a)^2 + y^2 = r^2$

Hence centre is $C(a, 0)$

$r = \sqrt{(a+9)^2 + 16}$ ...(1)

Also $\left|\frac{a-3}{\sqrt{2}}\right| = r$ ...(2)

$\sqrt{(a+9)^2 + 16} = \left|\frac{a-3}{\sqrt{2}}\right|$

$\Rightarrow$ $a = -5$ or -37

$r = \left|\frac{-5-3}{\sqrt{2}}\right|$ or $\left|\frac{-37-3}{\sqrt{2}}\right|$

$= 4\sqrt{2}$ or $20\sqrt{2}$

$\left|r_1^2 - r_2^2\right| = |32 - 800| = 768$

17. If $(I + A) = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & 0 \\ a & 2 & 2 \end{bmatrix}$, then the value of det $((a + 1)$ adj $((a - 1)A))$ is

(1) $4a^2(a-1)^3(a^2-1)^3$ (2) $8a^3(a^2-1)^6$

(3) $4a^2(a+1)^3(a^2+1)^3$ (4) $4a^2(a^2-1)^3(a+1)^3$

Answer (1)

Sol. $I + A = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & 0 \\ a & 2 & 2 \end{bmatrix}$

$\Rightarrow$ $A = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ a & 2 & 1 \end{bmatrix} \Rightarrow |A| = 2a$

det $((a + 1)$ adj $((a - 1)A))$

$= (a + 1)^3$ det (adj $(a - 1)A)$

$= (a + 1)^3$ det $((a - 1)A)^2$

$= (a + 1)^3 [(a - 1)^3]^2$ det $(A)^2$

$= (a - 1)^3 (a^2 - 1)^3 |A|^2$

$= 4a^2(a - 1)^3(a^2 - 1)^3$

18. Given $A = \{1, 2, ..., 40\}$. Three numbers are randomly selected from set A. Then, the probability that the terms form an increasing G.P is

(1) $\frac{1}{494}$ (2) $\frac{1}{247}$

(3) $\frac{1}{447}$ (4) $\frac{1}{397}$

Answer (1)

Sol. Total cases $= {}^{40}C_3 = 9880$

Since three number a, b and c are in G.P.

$\Rightarrow$ $b^2 = ac$

$\Rightarrow$ $b = \sqrt{ac}$

And this is an increasing G.P. therefore

If $a = 1$ $C = 4, 9, 16, 25, 36$

$a = 2$ $C = 8, 18, 32$

$a = 3$ $C = 12, 27$

$a = 4$ $C = 9, 16, 36$

$a = 5$ $C = 20$

$a = 6$ $C = 24$

$a = 7$ $C = 28$

$a = 8$ $C = 18, 32$

$a = 9$ $C = 36$

$a = 10$ $C = 40$

Total favourable cases = 20

Require probability $= \frac{20}{9880} = \frac{1}{494}$

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. The maximum value of n such that 50! is divisible by 3^n is

Answer (22)

Sol. $V_3(50!) = \left[\frac{50}{3}\right] + \left[\frac{50}{9}\right] + \left[\frac{50}{27}\right] + \left[\frac{50}{81}\right] + \cdots$

$= 16 + 5 + 1 + 0 + \cdots$

$= 22$

22. The total number of 10 digits sequences formed by only {0, 1, 2} where 1 should be used at least 5 times and 2 should be used exactly three times, is

Answer (2892)

Sol.

Zero	One	Two		
2	5	3	$\frac{10!}{2!5!3!} - \frac{9!}{5!3!1!}$	= 2520 – 504 = 2016
1	6	3	$\frac{10!}{1!6!3!} - \frac{9!}{6!3!}$	= 840 – 84 = 756
0	7	3	$\frac{10!}{9!3!}$	= 120

Total = 2016 + 756 + 120 = 2892

23. Let $f(x) = 2x^3 + 9x^2a + 12a^2x + 1$ has Local minima and local maxima occur at p & q respectively, such that $p^2 = q$. Then the value of $f(3)$ is

Answer (37)

Sol. $f(x) = 2x^3 + 9x^2a + 12a^2x + 1$

$f'(x) = 6x^2 + 18ax + 12a^2$

$f'(x) = 0$

$x^2 + 3ax + 2a^2 = 0$

$(x + 2a)(x + a) = 0$

$\Rightarrow x = -a, -2a$ [$a \neq 0$ as we will not get maxima and minima at $a = 0$]

Case 1 : When $a > 0$

$f''(x) = 12x + 18a$

$f''(-a) = -12a + 18a = 6a$

$f''(-2a) = -24a + 18a = -6a$

Minima at $x = -a$ & maxima at $x = -2a$

$p = -a$ & $q = -2a$

$p^2 = q$

$a^2 = -2a$

$a = 0, -2$

[Not possible]

Case II : When $a < 0$

$f''(-a) = 6a < 0$

$f''(-2a) = -6a > 0$

Maxima at $x = -a$

and Minima at $x = -2a$

$p = -2a,\ q = -a$

$p^2 = q$

$4a^2 = -a$

$4a^2 + a = 0$

$\Rightarrow \boxed{a = -\frac{1}{4}}$

$f(x) = 2x^3 + 9x^2\left(\frac{-1}{4}\right) + 12\left(\frac{-1}{4}\right)^2 x + 1$

$f(x) = 2x^3 - \frac{9x^2}{4} + \frac{3x}{4} + 1$

$f(3) = 54 - \frac{81}{4} + \frac{9}{4} + 1$

$\Rightarrow \boxed{f(3) = 37}$

24. If $\int_0^{e^2} \left[\frac{1}{e^{x-1}}\right] dx = \alpha - \log_e 2$, where [·] is Greatest Integer function, then α^3 equals to

Answer (8)

Sol. $\int_0^{e^2} \left[\frac{1}{e^{x-1}}\right] dx = \int_0^{e^2} \left[e^{1-x}\right] dx$

when, $x = 0$ then $[e^{1-x}] = [e] > 2$

when $e^{1-x} = 2$, then $x = 1 - \ln 2$

and when $e^{1-x} = 1$, then $x = 1$

Now $I = \int_0^{1-\ln 2} 2dx + \int_{1-\ln 2}^{1} 1dx + \int_1^{e^2} 0dx$

$= 2(1 - \ln 2) + (1 - (1 - \ln 2)$

$= 2 - 2\ln 2 + \ln 2$

$= 2 - \ln 2 = \alpha - \ln 2$

$\Rightarrow \alpha = 2$

$\alpha^3 = 8$

25.

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