MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. If the domain of the function $f(x)=\frac{1}{\sqrt{3x+10-x^2}}+\frac{1}{\sqrt{x+|x|}}$ is (a, b) then $(1+a)^2+b^2$ is equal to

(1) 25 (2) 16

(3) 24 (4) 26

Answer (4)

Sol. $x+|x|=\begin{cases}2x, & x\ge 0\\ 0, & x<0\end{cases}$

$\Rightarrow \frac{1}{\sqrt{x+|x|}}$, domain is $x > 0$, as $2x \neq 0$

Similarly,

$\frac{1}{\sqrt{3x+10-x^2}}$ is defined when $3x + 10 - x^2 > 0$

$\Rightarrow x^2 - 3x - 10 < 0$

$(x - 5)(x + 2) < 0$

$\Rightarrow x \in (-2, 5)$

$\Rightarrow$ Domain will be $(0, \infty) \cap (-2, 5) = (0, 5)$

$\Rightarrow (1 + a)^2 + b^2 = 1 + 25 = 26$

2. Find the eccentricity of the ellipse in which length of minor axis is equal to one-fourth of the distance between their foci.

(1) $\frac{4}{\sqrt{17}}$ (2) $\frac{2}{\sqrt{17}}$

(3) $\frac{7}{\sqrt{17}}$ (4) $\frac{8}{\sqrt{17}}$

Answer (1)

Sol. $2b=\frac{1}{4}(ae)\Rightarrow 4b=ae$

$b^2=a^2-a^2e^2$

$b^2=a^2-16b^2$

$17b^2=a^2$

$e=\sqrt{\frac{1-b^2}{a^2}}=\sqrt{\frac{1-1}{17}}=\frac{4}{\sqrt{17}}$

3. If two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=-\hat{i}+4\hat{j}+8\hat{k}$ and the vectors $\vec{c}$ and $\vec{d}$ are related as $(\vec{a}-\vec{c})\times\vec{b}=5\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}\times\vec{c}=\vec{d}$. Then $|\vec{a}\cdot\vec{d}|$ is equal to

(1) 12 (2) 8

(3) 10 (4) 7

Answer (3)

Sol. $(\vec{a}-\vec{c})\times\vec{b}=5\hat{i}-2\hat{j}+3\hat{k}$

$\vec{a}\times\vec{b}-\vec{c}\times\vec{b}=5\hat{i}-2\hat{j}+3\hat{k}$

$\vec{a}\times\vec{b}+\vec{d}=5\hat{i}-2\hat{j}+3\hat{k}$ (as $\vec{b}\times\vec{c}=\vec{d}$)

dot with $\vec{a}$

$\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{a}\cdot\vec{d}=\vec{a}\cdot(5\hat{i}-2\hat{j}+3\hat{k})$

$= 5 \times 1 + (-2)(2) + (3)(3)$

$= 5 - 4 + 9 = 10$

4. Evaluate $\int_{-2}^{2}\frac{9x^2}{1+5^x}dx$

(1) 12 (2) 24

(3) 30 (4) 15

Answer (2)

Sol. $I = \int_{-2}^{2} \frac{9x^2}{1+5^x}dx$ ...(1)

$I = \int_{-2}^{2} \frac{9x^2}{1+5^{-x}}dx$ ...(2)

Adding (1) and (2)

$2I = \int_{-2}^{2}\left(\frac{9x^2}{1+5^x} + \frac{5^x \cdot 9x^2}{1+5^x}\right)dx$

$2I = \int_{-2}^{2}\frac{9x^2}{1+5^x}\left(1+5^x\right)dx$

$2I = \int_{-2}^{2} 9x^2 dx$

$2I = 9\left(\frac{x^3}{3}\right)_{-2}^{2}$

$2I = 9\left(\frac{8}{3}+\frac{8}{3}\right) = 48$

$I = 24$

5. If the mean and variance of eight observations a, b, 8, 12, 10, 6, 4, 15, is 9 and 9.25 respectively. Then $a + b + ab$ is equal to

(1) 76

(2) 83

(3) 79

(4) 93

Answer (4)

Sol. Mean $= 9 = \frac{a+b+8+12+10+6+4+15}{8}$

$\Rightarrow$ $a + b + 55 = 72 \Rightarrow a + b = 17$

$\frac{a^2+b^2+64+144+100+36+16+225}{8} - 9^2 = 9.25$

$a^2 + b^2 + 585 - 8.9^2 = 74$

$\Rightarrow$ $a^2 + b^2 = 137$

$\Rightarrow$ $(a + b)^2 - 2ab = 137$

$\Rightarrow$ $2ab = 289 - 137 \Rightarrow ab = 76$

$\Rightarrow$ $a + b + ab = 17 + 76 = 93$

6. $4\int_0^1 \frac{1}{\sqrt{3+x^2}+\sqrt{1+x^2}}dx - 3\ln\sqrt{3}$ is equal to

(1) $3-\sqrt{2}+\ln\left(\sqrt{2}+1\right)$ (2) $2+\sqrt{2}-\ln\left(\sqrt{3}+1\right)$

(3) $2-\sqrt{2}-\ln\left(\sqrt{2}+1\right)$ (4) $2-\sqrt{3}-\ln\left(\sqrt{3}+1\right)$

Answer (3)

Sol. $I = 4\int_0^1 \frac{1}{\sqrt{3+x^2}+\sqrt{1+x^2}}dx$

$= 2\int_0^1 \sqrt{3+x^2}-\sqrt{1+x^2}\,dx$

$= 2\left[\int_0^1\sqrt{3+x^2}\,dx - \int_0^1\sqrt{1+x^2}\,dx\right]$

$= 2\left[\left(\frac{1}{2}x\sqrt{x^2+3}+\frac{3}{2}\ln\left|\sqrt{3+x^2}+x\right|\right) - \right.$

$\left.\left(\frac{1}{2}x\sqrt{1+x^2}+\frac{1}{2}\ln\left|\sqrt{1+x^2}+x\right|\right)\right]_0^1$

$= 2\left[\left(1+\frac{3}{2}\ln 3-\frac{3}{2}\ln\sqrt{3}\right)-\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\ln\left(\sqrt{2}+1\right)\right)\right]$

$= 2\left(1+\frac{3}{4}\ln 3-\frac{1}{\sqrt{2}}-\frac{1}{2}\ln\left(\sqrt{2}+1\right)\right)$

$= 3\ln\sqrt{3}+2-\sqrt{2}-\ln\left(\sqrt{2}+1\right)$

$I - 3\ln\sqrt{3} = 2-\sqrt{2}-\ln\left(\sqrt{2}+1\right)$

7. If $y = \cos\left(\frac{\pi}{3}+\cos^{-1}\left(\frac{x}{2}\right)\right)$, then which of the following is true.

(1) $x^2 - 2xy + 8y^2 = 2$

(2) $x^2 - 2xy + 4y^2 = 3$

(3) $x^2 - 3xy + 4y^2 = 3$

(4) $x^2 - 5xy + 4y^2 = 8$

Answer (2)

Sol. $\therefore y = \cos\left(\frac{\pi}{3}+\cos^{-1}\frac{x}{2}\right)$

$y = \cos\frac{\pi}{3}\cdot\cos\left(\cos^{-1}\frac{x}{2}\right) - \sin\frac{\pi}{3}\cdot\sin\left(\cos^{-1}\frac{x}{2}\right)$

$y = \frac{1}{2}\cdot\frac{x}{2} - \frac{\sqrt{3}}{2}\cdot\sqrt{1-\frac{x^2}{4}}$

$4y = x - \sqrt{3}\sqrt{4-x^2}$

$(4y-x)^2 = 3(4-x^2)$

$16y^2 + x^2 - 8xy = 12 - 3x^2$

$4x^2 - 8xy + 16y^2 = 12$

$\therefore\ x^2 - 2xy + 4y^2 = 3$

8. The image of the point (1, 0, 3) about the line passing through $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ and whose direction ratios are $\vec{r} = 4\hat{i} + 2\hat{j} - \hat{k}$ is

(1) $\left(\frac{-23}{21}, \frac{20}{21}, \frac{-73}{21}\right)$ (2) $\left(\frac{1}{21}, \frac{-23}{21}, \frac{-31}{21}\right)$

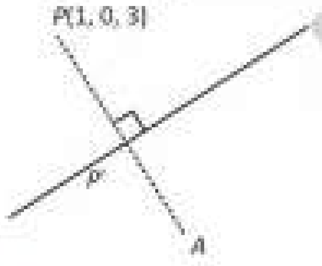
(3) $\left(\frac{1}{21}, \frac{21}{23}, \frac{-30}{21}\right)$ (4) $\left(\frac{3}{21}, \frac{7}{21}, \frac{-5}{21}\right)$

Answer (1)

Sol. $\vec{a} : 3\hat{i} + 2\hat{j} - \hat{k}$

Dr: $4\hat{i} + 2\hat{j} - \hat{k}$

$L : \frac{x-3}{4} = \frac{y-2}{2} = \frac{z+1}{-1}$



Any point on line L: P′ (4λ +3, 2λ + 2, -λ -1)

$PP' \cdot \vec{n} = 0$

$\Rightarrow 4(4\lambda+2) + 2(2\lambda+2) + (-\lambda-4)(-1) = 0$

$16\lambda + 8 + 4\lambda + 4\lambda + 4 = 0$

$21\lambda + 16 = 0$

$\lambda = \frac{-16}{21}$

$\therefore\ P'\left(\frac{-1}{21}, \frac{10}{21}, \frac{-5}{21}\right)$

Let image of point P be (a, b, c)

$\therefore\ \frac{a+1}{2} = \frac{-1}{21} \Rightarrow a = \frac{-23}{21}$

$\frac{b+0}{2} = \frac{10}{21} \Rightarrow b = \frac{20}{21}$

$\frac{c+3}{2} = \frac{-5}{21} \Rightarrow c = \frac{-73}{21}$

$\therefore$ image will be $\left(\frac{-23}{21}, \frac{20}{21}, \frac{-73}{21}\right)$

9. If the curve $x^2 = 4y$ intersects the line $y = 2(x + 6)$ at (a, b) in 2nd quadrant, then $\int_a^b \frac{x^4}{1+5^x}\,dx$ is

(1) $\frac{512}{5}$ (2) $\frac{1024}{5}$

(3) $\frac{32}{5}$

(4) $\frac{16}{5}$

Answer (2)

Sol. $x^2 = 4y$

$y = 2(x + 6)$

$x^2 = 8(x + 6)$

$x^2 - 8x - 48 = 0$

$(x + 4)(x - 12) = 0$

$\Rightarrow\ x = -4\ (\because x < 0)$

$\therefore\ y = 4$

$\Rightarrow\ (a, b) = (-4, 4)$

$I = \int_a^b \frac{x^4}{1+5^x}\,dx$

$= \int_{-4}^{4} \frac{x^4}{1+5^x}\,dx$

$$= \int_0^4 \left(\frac{x^4}{1+5^x} + \frac{x^4}{1+5^{-x}} \right) dx$$

$$= \int_0^4 x^4 dx = \frac{x^5}{5} = \frac{4^5}{5} = \frac{1024}{5}$$

10. If $\lim_{x \to 0} \frac{\cos 2x + a\cos^4 x - b}{x^4} = L$ (finite)

then $a + b$ equals to

(1) −1 (2) 0

(3) 2 (4) 3

Answer (1)

Sol. $\lim_{x \to 0} \frac{\cos 2x + a\cos^4 x - b}{x^4} = L$

$\lim_{x \to 0} \frac{2\cos^2 x - 1 + a\cos^4 x - b}{x^4} = L$ ...(1)

To get the finite value,

$1 + a - b = 0$

$\Rightarrow a = b - 1$ ...(2)

Apply L Hospital

$\lim_{x \to 0} \frac{4\cos x(-\sin x) + 4a\cos^3 x(-\sin x)}{4x^3}$

$\lim_{x \to 0} \frac{4\cos x + 4a\cos^3 x}{4x^2}\left(\frac{-\sin x}{x}\right)$

To get the finite value, $a = -1$

Also from (1)

$b = 0$

$\therefore a + b = -1$

11. If the sum of series $\frac{1}{1+4.1^4} + \frac{2}{1+4.2^4} + \frac{3}{1+4.3^4} + ... + \frac{10}{1+4.10^4}$ is $\frac{m}{n}$, where m and n are natural coprime numbers, then $(m + n)$ is

(1) 289 (2) 276

(3) 225 (4) 389

Answer (2)

Sol. $T_r = \frac{r}{1+4 \cdot r^4} = \frac{r}{4r^4 + 4r^2 + 1 - 4r^2} = \frac{r}{(2r^2+1)^2 - (2r)^2} = \frac{r}{(2r^2 - 2r + 1)(2r^2 + 2r + 1)}$

$$T_r = \frac{1}{4}\left[\frac{(2r^2+2r+1) - (2r^2-2r+1)}{(2r^2-2r+1)(2r^2+2r+1)}\right]$$

$$= \frac{1}{4}\left[\frac{1}{r^2+(r-1)^2} - \frac{1}{r^2+(r+1)^2}\right]$$

$$\sum_{r=1}^{10} T_r = \frac{1}{4}\left[\frac{1}{0^2+1^2} - \frac{1}{1^2+2^2} + \frac{1}{1^2+2^2} - \frac{1}{2^2+3^2} + \frac{1}{9^2+10^2} - \frac{1}{10^2+11^2}\right]$$

$$= \frac{1}{4}\left[\frac{1-1}{221}\right] = \frac{220}{4 \times 221} = \frac{55}{221}$$

12. A bag is Randomly selected, If drawn ball is red, then probability that ball is selected from bag-I is p. If ball drawn is green then probability that ball is selected from bag-III is q. Then $\frac{1}{p} + \frac{1}{q}$ equals to

	Red	Blue	Green
Bag-I	3	3	4
Bag-II	4	3	3
Bag-III	5	2	3

(1) $\frac{22}{3}$ (2) $\frac{22}{5}$

(3) $\frac{11}{3}$ (4) $\frac{11}{5}$

Answer (1)

Sol. $p(B_1 / R) = \frac{p(B_1) \cdot p(R / B_1)}{p(R)}$

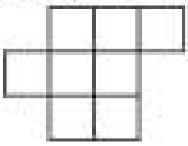
$$= \frac{\frac{1}{3} \times \frac{3}{10}}{\frac{1}{3} \times \frac{3}{10} + \frac{1}{3} \times \frac{4}{10} + \frac{1}{3} \times \frac{5}{10}} = \frac{1}{4} = p$$

$p(B_3 / G) = \dfrac{p(B_3)\cdot p(G / B_3)}{p(G)}$

$$= \frac{\frac{1}{3}\times\frac{3}{10}}{\frac{1}{3}\times\frac{3}{10}+\frac{1}{3}\times\frac{3}{10}+\frac{1}{3}\times\frac{4}{10}} = \frac{3}{10} = q$$

$\dfrac{1}{p}+\dfrac{1}{q} = 4+\dfrac{10}{3} = \dfrac{22}{3}$

13. In the given figure, number of ways to fill *a, b, c, d* and *e* into boxes such that no row is empty and at most one letter is filled in one box, is



(1) 5670

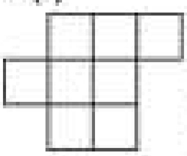
(2) 5760

(3) 5880

(4) 720

Answer (2)

Sol.



Let *x, y, z* be the number of box which are filled

$\Rightarrow\ 1 \le x \le 3, 1 \le y \le 3, 1 \le z \le 2$

x	y	z	Number of ways
3	1	1	${}^3C_3 \cdot {}^3C_1 \cdot {}^2C_1 = 6$
2	2	1	${}^3C_2 \cdot {}^3C_2 \cdot {}^2C_1 = 18$
1	3	1	${}^3C_1 \cdot {}^3C_3 \cdot {}^2C_1 = 6$
2	1	2	${}^3C_2 \cdot {}^3C_1 \cdot {}^2C_2 = 9$
1	2	2	${}^3C_1 \cdot {}^3C_2 \cdot {}^2C_2 = 9$

Total ways = (48) to fill boxes

Now to arrange *a, b, c, d* and *e*

Number of ways will be 48.5! = 5760

14.

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. Total number of terms in an AP are even. Sum of odd terms is 24 and sum of even terms is 30. Last term exceeds the first term by $\dfrac{21}{2}$. Then the total number of terms is

Answer (8)

Sol. Let the number of terms be 2n

$$T_1 + T_3 + T_5 \ldots T_{2n-1} = 24$$
$$T_2 + T_4 + T_6 \ldots T_{2n} = 30$$
$$(T_2 - T_1) + (T_4 - T_3) + \ldots(T_{2n} - T_{2n-1}) = 6$$

nd = 6

$(a + (2n + 1)d) - a = \dfrac{21}{2}$

$\Rightarrow\ 2nd - d = \dfrac{21}{2}$

$\Rightarrow\ 12 - \dfrac{21}{2} = d$

$\Rightarrow\ d = \dfrac{3}{2}$

$\therefore\ n = 4$

$\therefore$ Total terms = 8

22. If $\dfrac{dy}{dx} + 2y\sec^2x = 2\sec^2x + 3\tan x\sec^2x$ and $f(0) = \dfrac{5}{4}$. Then the value of $12\left(y\left(\dfrac{\pi}{4}\right) - \dfrac{1}{e^2}\right)$ equal to

Answer (21)

Sol. $\frac{dy}{dx} + 2y\sec^2 x = 2\sec^2 x + 3\tan x\sec^2 x$

$\text{I.F.} = e^{\int 2\sec^2 x\,dx}$

$\text{I.F.} = e^{2\tan x}$

$y \cdot e^{2\tan x} = \int e^{2\tan x}(2 + 3\tan x)\sec^2 x\,dx$

Put $\tan x = u$

$\sec^2 x\,dx = du$

$y \cdot e^{2u} = \int e^{2u}(2 + 3u)\,du$

$y \cdot e^{2u} \Rightarrow \frac{2e^{2u}}{2} + 3\int e^{2u} \cdot u\,du$

$y \cdot e^{2u} = e^{2u} + 3\left[\frac{ue^{2u}}{2} - \int \frac{e^{2u}}{2}\right]$

$ye^{2u} = e^{2u} + 3\left[\frac{ue^{2u}}{2} - \frac{e^{2u}}{4}\right] + C$

$ye^{2\tan x} = e^{2\tan x} + 3\left[\frac{\tan x e^{2\tan x}}{2} - \frac{e^{2\tan x}}{4}\right] + C$

$F(0) = \frac{5}{4}$

$\frac{5}{4} = 1 - \frac{3}{4} + C$

$\frac{5}{4} - \frac{1}{4} = C$

$1 = C$

$y = 1 + 3\left(\frac{\tan x}{2} - \frac{1}{4}\right) + 1 \cdot e^{-2\tan x}$

$y\left(\frac{\pi}{4}\right) = 1 + 3\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{e^2}$

$y\left(\frac{\pi}{4}\right) = \frac{7}{4} + \frac{1}{e^2}$

$12\left(y\left(\frac{x}{4}\right) - \frac{1}{e^2}\right) = 12\left(\frac{7}{4} + \frac{1}{e^2} - \frac{1}{e^2}\right) = 21$

23. If the non-zero 3 × 3 matrix A satisfies

$A^2(A - 4I) - 4(A - I) = 0$ and if $A^5 = \alpha A^2 + \beta A + \gamma I$, where I is 3 × 3 identity matrix, then $\alpha + \beta + \gamma$ is equal to

Answer (76)

Sol. $A^2(A - 4I) - 4(A - I) = 0$

$A^3 - 4A^2 - 4A + 4I = 0$

Multiple by A

$A^4 = 4A^3 + 4A^2 - 4A$

$= 4(4A^2 + 4A - 4I) + 4A^2 - 4A$

$= 20A^2 + 12A - 16I$

Multiple again by A

$\Rightarrow A^5 = 20A^3 + 12A^2 - 16A$

$= 20(4A^2 + 4A - 4I) + 12A^2 - 16A$

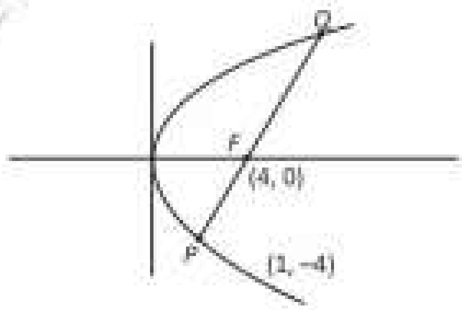
$= 92A^2 + 64A - 80I = \alpha A^2 + \beta A + \gamma I$

$\Rightarrow \alpha = 92, \beta = 64, \gamma = -80 \Rightarrow \alpha + \beta + \gamma = 76$

24. If PQ be the focal chord of a parabola $y^2 = 16x$ such that $P(1, -4)$ and $\frac{PF}{QF} = \frac{m}{n}$, ($F$ is focus) where m and n are coprime natural numbers, then $m^2 + n^2$ is

Answer (17)

Sol.



$y^2 = 16x$

$\Rightarrow 4a = 16 \Rightarrow a = 4$

$Q \equiv (at_2^2, 2at_2)$

$\equiv (4t_2^2, 8t_2)$

$P = \left(4t_1^2, 8t_1\right)$

$4t_1^2 = 1,\ 8t_1 = -4 \Rightarrow t_1 = \frac{-1}{2}$

since P and Q are ends points of focal chord

$t_1t_2 = 1 \Rightarrow t_2 = 2$

$\Rightarrow\ Q = (16, 16)$

$\Rightarrow\ PF = \sqrt{3^2 + 4^2},\ FQ = \sqrt{12^2 + 16^2}$

$\Rightarrow\ \frac{PF}{QF} = \frac{5}{20} = \frac{1}{4} = \frac{m}{n}$

$\Rightarrow\ m^2 + n^2 = 17$

25.

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