MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Solve $\int_{-1}^{1}\frac{1+2x}{e^{-x}+e^{x}}dx$

   (1) $2\left(\tan^{-1}e-\frac{\pi}{4}\right)$ (2) $2\left(\tan^{-1}e-\frac{\pi}{3}\right)$

   (3) $2\left(\tan^{-1}e-\frac{\pi}{2}\right)$ (4) $2\left(\frac{\pi}{2}-\tan^{-1}e\right)$

Answer (1)

Sol. $I=\int_{-1}^{1}\frac{1+2x}{e^{x}+e^{-x}}dx$

$I=\int_{0}^{1}\frac{(1+2x)+(1-2x)}{e^{x}+e^{-x}}dx$

$=\int_{0}^{1}\frac{2e^{x}}{e^{2x}+1}dx$

Let $e^x = t$

$=\int_{1}^{e}\frac{2dt}{t^2+1}$

$=2\left[\tan^{-1}t\right]_1^e$

$=2\left(\tan^{-1}e-\frac{\pi}{4}\right)$

Prop. $\int_{-a}^{a}f(x)dx=\int_{0}^{a}(f(x)+f(-x))dx$

2. The sum of the series 1 + 3 + 5^2 + 7 + 9^2 + ... upto 80 terms is

   (1) 328160

   (2) 338160

   (3) 339400

   (4) 326870

Answer (2)

Sol. 1 + 3 + 5^2 + 7 + 9^2 + ...

= (1^2 + 5^2 + 9^2 + ...) + (3 + 7 + 11+...)

$=\left(\sum_{k=1}^{40}(4k-3)^2\right)+\frac{40}{2}(6+(40-1)4)$

$=16\sum_{k=1}^{40}k^2+9\times40-24\sum_{k=1}^{40}k+3240$

$=16\left(\frac{40\times41\times81}{6}\right)+360-24\left(\frac{40\times41}{2}\right)+3240$

= 354240 + 360 – 19680 + 3240 = 338160

3. Let there be two A.P.'s with each having 2025 terms. Find the number of distinct terms in union of these two A.P.'s, i.e., $A\cup B$ if first A.P. is 1, 6, 11, ... and second A.P. is 9, 16, 23, ...

   (1) 3761 (2) 4035

   (3) 3022 (4) 2025

Answer (1)

Sol. 1st A.P. : 1, 6, 11... $\Rightarrow T_n = S_n - 4$

2nd A.P.: 9, 16, 23... $\Rightarrow T_m = 2 + 7m$

Let's find when they are equal for the first time:

$5n - 4 = 2 + 7m$

$\Rightarrow$ $5n - 7m = 6$

$\Rightarrow$ $n = 4, m = 2$

$\Rightarrow$ 16 is the first term, common difference will be

LCM (d_1, d_2) = LCM (5, 7) = 35

$\Rightarrow$ Common terms will be 16, 51, 86 ...

The last term of 1st A.P.

= T_{2025} = 5 × 2025 – 4 = 10121

$\Rightarrow$ Common term must be less than that

$\Rightarrow$ $35n - 19$

$\Rightarrow 35n - 19 \le 10121 \Rightarrow 35n \le 10140$

$\Rightarrow n \le 289.7$

$\Rightarrow \boxed{n = 289}$

$\Rightarrow$ in $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

= 2025 + 2025 – 289

= 3761

4. Consider a committee of 12 members is formed randomly out of 4 Engineers, 2 Doctors and 10 Professors. Find the probability that the committee has exactly 3 Engineers and 1 Doctor.

(1) $\frac{17}{91}$ (2) $\frac{18}{91}$

(3) $\frac{15}{91}$ (4) $\frac{18}{71}$

Answer (2)

Sol. $P(3E, 1D) = \dfrac{{}^{4}C_3 \cdot {}^{2}C_1 \cdot {}^{10}C_8}{{}^{16}C_{12}}$

$= \dfrac{4 \times 2 \times \frac{10 \times 9}{2}}{\frac{16 \times 15 \times 14 \times 13}{24}} = \dfrac{360 \times 24}{13 \times 7 \times 30 \times 16}$

$= \dfrac{18}{91}$

Explanation: 3 Engineers, 1 Doctors and remaining from Professors.

5. The number of integral values of $n \in N$ for which the equation $x^2 + 4x - n = 0$, $n \in [20, 100]$ have integral roots, is

(1) 4 (2) 5

(3) 6 (4) 7

Answer (3)

Sol. $(x+2)^2 = n + 4 \Rightarrow x = -2 \pm \sqrt{n+4}$

$\because$ $20 \le n \le 100 \Rightarrow 24 \le n + 4 \le 104$

$\Rightarrow$ $\sqrt{24} \le \sqrt{n+4} \le \sqrt{104}$

$\sqrt{n+4} \in \{5, 6, 7, 8, 9, 10\}$

$\Rightarrow$ For integral values of x,

$\Rightarrow$ $n = \{5^2 - 4, 6^2 - 4, 7^2 - 4, 8^2 - 4, 9^2 - 4, 10^2 - 4\}$

$\Rightarrow$ There are six integral values of n.

6. If $10\sin^4\theta + 15\cos^4\theta = 6$, then find the value of

$\dfrac{27\text{cosec}^6\theta + 8\sec^6\theta}{8\sec^8\theta}$

(1) $\frac{3}{5}$ (2) $\frac{2}{5}$

(3) $\frac{1}{5}$ (4) $\frac{4}{5}$

Answer (1)

Sol. $10(\sin^4\theta + \cos^4\theta) + 5\cos^4\theta = 6$

$\Rightarrow$ $10[1 - 2\sin^2\theta\cos^2\theta] + 5\cos^4\theta = 6$

$\Rightarrow$ $25\cos^4\theta - 20\cos^2\theta + 4 = 0$

$\Rightarrow$ $(5\cos^2\theta - 2)^2 = 0$

$\therefore$ $\cos^2\theta = \frac{2}{5}$

$\Rightarrow$ $\sin^2\theta = \frac{3}{5}$ and $\sec^2\theta = \frac{5}{2}, \text{cos}ec^2\theta = \frac{5}{3}$

$\Rightarrow$ $\dfrac{27\text{cosec}^6\theta + 8\sec^6\theta}{8\sec^8\theta} = \dfrac{27\cdot\left(\frac{5}{3}\right)^3 + 8\cdot\left(\frac{5}{2}\right)^3}{8\cdot\left(\frac{5}{4}\right)^4} = \dfrac{4}{5}$

7. Find the length of latus rectum of an ellipse if foci are (2, 5) and (2, –3) and the eccentricity of the ellipse is $\frac{4}{5}$

(1) $\frac{32}{3}$ (2) $\frac{32}{5}$

(3) $\frac{18}{5}$ (4) $\frac{16}{5}$

Answer (3)

Sol. F_1 : (2, 5) and F_2 : (2, –3), notice major axis along *y*-axis

$$\Rightarrow F_1F_2 = 8 = 2be \Rightarrow b = \frac{8}{2e} = \frac{4}{4/5} = 5$$

$$\Rightarrow e^2 = 1 - \frac{a^2}{b^2} = 1 - \frac{a^2}{25} = \frac{16}{25}$$

$$\Rightarrow a^2 = 9 \Rightarrow a = 3$$

The length of latus rectum : $\frac{2a^2}{b} = \frac{2(9)}{5} = \frac{18}{5}$

8. If $\lim_{x\to1^+} \frac{(x-1)[6+\lambda\cos(x-1)]+\mu\sin(x-1)}{(x-1)^3} = -1$.

Then the value of $\lambda + \mu$ is

(1) –6

(2) –4

(3) 4

(4) 2

Answer (1)

Sol. $\lim_{x\to1^+} \frac{(x-1)[6+\lambda\cos(x-1)]+\mu\sin(x-1)}{(x-1)^3} = -1$

$$\lim_{x\to1^+} \frac{(x-1)\left[6+\lambda\cos\left(1-\frac{(x-1)^2}{2!}+\frac{(x-1)^4}{4!}\right)\right]+\mu\left((x-1)-\frac{(x-1)^3}{3!}+...\right)}{(x-1)^3}$$

$$\lim_{x\to1^+} \frac{(x-1)(6+\lambda)-\frac{\lambda(x-1)^3}{2!}+...+\mu(x-1)-\mu\frac{(x-1)^3}{3!}+..}{(x-1)^3}$$

For limit to exist

$$6+\lambda+\mu=0$$

$$\Rightarrow \boxed{\lambda+\mu=-6}$$

9. Let α and β be the number of points where the function. $f(x)$ = max{x, x^3, x^5, ... x^{21}} is not conations and not differentiable respectively, then $\alpha + \beta$ equals to

(1) 6 (2) 3

(3) 2 (4) 4

Answer (2)

Sol. For $x \ge 1$, $x^{21} \ge x^{19} \ge .... \ge x$

$$\Rightarrow f(x) = \begin{cases} x & x<-1 \\ x^{21} & -1\le x\le 0 \\ x & 0<x<1 \\ x^{21} & x\ge 1 \end{cases}$$

Closely, $f(x)$ is continuous everywhere $\Rightarrow \alpha = 0$

$$f'(x) = \begin{cases} 1 & , \quad x<-1 \\ 20x^{20} & , \quad -1\le x\le 0 \\ 1 & , \quad 0<x<1 \\ 21x^{20} & , \quad x\ge 1 \end{cases}$$

$\Rightarrow \beta = 3$

$\Rightarrow \alpha + \beta = 3$

10. If $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$, then the area bounded by the curve $y = f(x)$ and coordinate axes is (in square units)

(1) $\frac{1}{2}$ (2) 1

(3) $\frac{3}{2}$ (4) 2

Answer (1)

Sol. $\because f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$

or, $f(x) = 1 - 2x + e^x \int_0^x e^{-t} f(t)dt$

on differentiating both sides w.r.t. *x* we get

$$f'(x) = -2 + e^x \int_0^x e^{-t} f(t)dt + e^x \cdot e^{-x} f(x)$$

$$f'(x) = -2 + f(x) + 2x - 1 + f(x) \quad \text{\{from eq. (1)\}}$$

$\therefore$ $f'(x) - 2f(x) = 2x - 3$

I.F. = $e^{\int -2dx} = e^{-2x}$

$$\therefore \quad e^{-2x} \cdot f(x) = \int e^{-2x}(2x-3)dx$$

$$e^{-2x} \cdot f(x) = (2x-3)\cdot\frac{e^{-2x}}{-2} - \int 2\cdot\frac{e^{-2x}}{-2}dx$$

$$e^{-2x} \cdot f(x) = \frac{(2x-3)e^{-2x}}{-20} + \frac{e^{-2x}}{-2} + c$$

$f(x) = -x + 1 + c'e^{2x}$

$\because$ $f(x) = 1$ from eq. (1)

$\therefore$ $1 = 0 + 1 + c' \Rightarrow c' = 0$

$\therefore$ $f(x) = -x + 1$

$\Rightarrow$ Area = $\frac{1}{2}$

11. The value of $\sin^{-1}\left(\frac{\sqrt{3}x}{2} + \frac{1}{2}\sqrt{1-x^2}\right), -\frac{1}{2} < x < \frac{1}{\sqrt{2}}$ is equivalent to

(1) $\frac{2\pi}{3} - \cos^{-1}x,\ -\frac{1}{2} < x < \frac{1}{\sqrt{2}}$

(2) $\pi - \cos^{-1}x,\ -\frac{1}{2} < x < \frac{1}{\sqrt{2}}$

(3) $\frac{\pi}{3} - \cos^{-1}x,\ -\frac{1}{2} < x < \frac{1}{\sqrt{2}}$

(4) $\frac{\pi}{2} - \sin^{-1}x,\ -\frac{1}{2} < x < \frac{1}{\sqrt{2}}$

Answer (1)

Sol. $\sin^{-1}\left(\frac{\sqrt{3}x}{2} + \frac{1}{2}\sqrt{1-x^2}\right), -\frac{1}{2} < x < \frac{1}{\sqrt{2}}$

Let $x = \cos\theta,\ \theta \in \left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow$ $\sqrt{1-x^2} = \sin\theta$, as $\sin\theta > 0$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\cos\theta + \cos\frac{\pi}{3}\cos\theta\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{3} + \theta\right)\right)$$

Since $\frac{\pi}{3} + \theta \in \left(\frac{7\pi}{12}, \pi\right)$

$\Rightarrow$ $\sin\left(\frac{\pi}{3} + \theta\right) = \sin\left(\pi - \left(\frac{\pi}{3} + \theta\right)\right) = \sin\left(\frac{2\pi}{3} - \theta\right)$

$\Rightarrow$ $\sin^{-1}\left(\sin\left(\frac{\pi}{3} + \theta\right)\right) = \sin^{-1}\left(\sin\left(\frac{2\pi}{3} - \theta\right)\right)$

Since $\frac{2\pi}{3} - \theta \in \left(0, \frac{5\pi}{12}\right)$

$\Rightarrow$ $= \frac{2\pi}{3} - \theta$

$= \frac{2\pi}{3} - \cos^{-1}x,\ x \in \left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

12. There are 10 pens such that 3 pens are defective. Let X represent the number of defective pen selected. If two pens are selected at random then variance of X is

(1) $\frac{38}{75}$

(2) $\frac{28}{75}$

(3) $\frac{14}{75}$

(4) $\frac{3}{75}$

Answer (2)

Sol.

X	P(X)	XP(X)	$(X_i - \mu)^2$	$P_iX(X_i - \mu)^2$
$X = 0$	$\frac{^7C_2}{^{10}C_2}$	0	$\left(0 - \frac{3}{5}\right)^2$	$\frac{7}{15}\left(\frac{9}{25}\right)$
$X = 1$	$\frac{^7C_1\ ^3C_1}{^{10}C_2}$	$\frac{7}{15}$	$\left(1 - \frac{3}{5}\right)^2$	$\frac{7}{15}\left(\frac{4}{25}\right)$
$X = 2$	$\frac{^7C_0\ ^3C_2}{^{10}C_2}$	$\frac{2}{15}$	$\left(2 - \frac{3}{5}\right)^2$	$\frac{1}{15}\left(\frac{49}{25}\right)$

$\mu = \sum X_iP(X_i) = 0 + \frac{7}{15} + \frac{2}{15} = \frac{3}{5}$

Variance (X) =

$$\sum P_i(X_i - \mu)^2 = \frac{7}{15}\left(\frac{9}{25}\right) + \frac{7}{15}\left(\frac{4}{25}\right) + \frac{1}{15}\left(\frac{49}{25}\right) = \frac{28}{75}$$

13.

14.

15.

16.

17.

18.

19.

20.

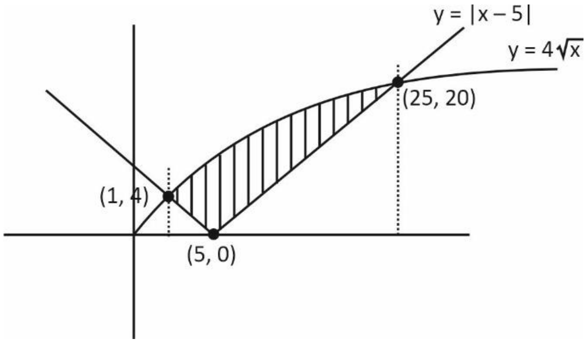
SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. Let $|x-5| \le y \le 4\sqrt{x}$. If the Area enclosed is *A,* then 3*A* equals to

Answer (368)

Sol.



$$A = \int_1^{25} 4\sqrt{x}dx - \frac{1}{2}\times 4\times 4 - \frac{1}{2}\times 20\times 20$$

$$A = \left[\frac{4x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^{25} - 8 - 200$$

$$A = \frac{8}{3}[125-1] - 208$$

$$= \frac{368}{3} \Rightarrow 3A = 368$$

22. Given two lines

$$L_1 : \frac{x-3}{3} = \frac{y-\alpha}{1} = \frac{z+2}{-2}$$

and $L_2 : \frac{x+1}{2} = \frac{y+2}{1} = \frac{z-\beta}{-1}$

If shortest distance between L_1 and L_2 is $30\sqrt{3}$. Then the value of |a + b| is

Answer (90)

Sol. $L_1 : \frac{x-3}{3} = \frac{y-\alpha}{1} = \frac{z+2}{-2}$

$L_2 : \frac{x+1}{2} = \frac{y+2}{1} = \frac{z-\beta}{-1}$

Shortest distance, $d = \frac{\left|(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}-\vec{b_2})\right|}{\left|(\vec{b_1}\times\vec{b_2})\right|}$

$$(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}-\vec{b_2}) = \begin{vmatrix} 4 & \alpha+2 & -2-\beta \\ 3 & 1 & -2 \\ 2 & 1 & -1 \end{vmatrix}$$

= 4 (– 1 + 2) – (α + 2) (–3 + 4)

– (2 + β) (3 – 2)

= 4 – α – 2 – 2 – β

= – α – β

Also $|\vec{b_1}\times\vec{b_2}| = \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 2 & 1 & -1 \end{matrix}\right\|$

$= |\hat{i} - \hat{j} + \hat{k}|$

$= \sqrt{3}$

$30\sqrt{3} = \frac{|-\alpha-\beta|}{\sqrt{3}}$

$\Rightarrow \quad |\alpha+\beta| = 90$

23. If $\vec{v} = 2\hat{i} + \hat{j} - \lambda\hat{k}$, $(\lambda > 0)$, $\vec{u} = 3\hat{i} - \hat{j}$ and $\vec{v}_1$ is parallel to $\vec{u}$, $\vec{v}_2$ is perpendicular to $\vec{u}$ and $\vec{v} = \vec{v}_1 + \vec{v}_2$. If angle between $\vec{v}$ and $\vec{v}_1$ is $\cos^{1}\left(\frac{\sqrt{5}}{2\sqrt{7}}\right)$, then $|\vec{v}_1|^2 + |\vec{v}_2|^2$ equals to

Answer (14)

Sol. $\vec{v} = \vec{v}_1 + \vec{v}_2$

$\vec{v}\cdot\vec{u} = \vec{v}_1\cdot\vec{u} + \vec{v}_2\cdot\vec{u} = (k\vec{u})\cdot\vec{u} + 0$

6 – 1 + 0 = *k*(0) $\quad \left[\begin{matrix}\vec{v}_1 \parallel \vec{u} \\ \vec{v}_2 \perp \vec{u}\end{matrix}\right] \Rightarrow \vec{v}_1 = k\vec{u}$

$\Rightarrow \quad k = \frac{1}{2}$

$\vec{v}_1 = \frac{1}{3}(3\hat{i} - \hat{j}) = \frac{3}{2}\hat{i} - \frac{\hat{j}}{2}$

Cosine of angle between $\vec{v}$ and $\vec{v_1}$

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$$= \frac{v\cdot\vec{v}_1}{|\vec{v}||\vec{v}_1|} = \frac{3-\frac{1}{2}+0}{\sqrt{\frac{9}{4}+\frac{1}{4}}\sqrt{4+1+\lambda^2}}$$

$$\Rightarrow \quad \frac{\sqrt{5}}{2\sqrt{7}} = \frac{\frac{5}{2}}{\sqrt{\frac{5}{2}}\sqrt{\lambda^2+5}}$$

$$\Rightarrow \quad \frac{5}{4\times 7} = \frac{\frac{5}{2}}{(\lambda^2+5)} \Rightarrow (\lambda^2+5) = 14$$

$\lambda^2 = 9$, $\lambda = 3$

$$\vec{v_2} = \vec{v} - \vec{v}_1 = (2\hat{i}+\hat{j}-3\hat{k}) - \left(\frac{3}{2}\hat{i} - \frac{\hat{j}}{2}\right)$$

$$= \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$$

$$|\vec{v}_1|^2 + |\vec{v}_2|^2 = \left(\frac{9}{4}+\frac{1}{4}\right) + \left(\frac{1}{4}+\frac{9}{4}+9\right)$$

$$= \frac{10}{4} + \frac{10}{4} + 9 = 14$$

24.

25.

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