

Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.
2. **Section A** contains a total of 30 **Multiple Choice Questions (MCQ)**. Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.
3. **Section B** contains a total of 10 **Multiple Select Questions (MSQ)**. Each MSQ type question is similar to MCQ but with a difference that there will be **one or more than one** choices that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.
4. **Section C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.
5. In all sections, questions not attempted will result in zero mark. In **Section A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1-mark questions, $\frac{1}{3}$ marks will be deducted for each wrong answer. For all 2-mark questions, $\frac{2}{3}$ marks will be deducted for each wrong answer. In **Section B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section C** (NAT) as well.
6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.
7. A Scribble Pad will be provided for rough work.

Special Instructions / Useful Data	
\mathbb{N}	The set of positive integers
\mathbb{R}	The set of real numbers
\mathbb{R}^n	$\{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \dots, n\}$
$\ln x$	Natural logarithm of x
$\mathbb{P}(E)$	Probability of event E
$\mathbb{P}(E F)$	Conditional probability of event E given the occurrence of event F
$\mathbb{E}(X)$	Expectation of a random variable X
$\text{Var}(X)$	Variance of a random variable X
$\text{Cov}(X, Y)$	Covariance between the random variables X and Y
$\text{Bin}(n, p)$	Binomial distribution with parameters n and p ; $n \in \mathbb{N}, 0 < p < 1$
$\text{Poisson}(\lambda)$	Poisson distribution with mean λ ; $\lambda > 0$
$U(a, b)$	Continuous uniform distribution on the interval (a, b) ; $a < b, a, b \in \mathbb{R}$
$\text{Exp}(\lambda)$	Exponential distribution with mean $1/\lambda$; $\lambda > 0$
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2 ; $\mu \in \mathbb{R}, \sigma > 0$
$N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$	Bivariate normal distribution with means μ_1, μ_2 , variances σ_1^2, σ_2^2 and correlation ρ ; $\mu_1 \in \mathbb{R}, \mu_2 \in \mathbb{R}, \sigma_1 > 0, \sigma_2 > 0, -1 < \rho < 1$
$\Phi(\cdot)$	The cumulative distribution function of the $N(0, 1)$ random variable
χ_n^2	Central chi-square distribution with n degrees of freedom
t_n	Central Student's t distribution with n degrees of freedom
$F_{m,n}$	Central F distribution with m and n degrees of freedom
$\chi_{n,\alpha}^2$	A constant such that $\mathbb{P}(X > \chi_{n,\alpha}^2) = \alpha$, where $X \sim \chi_n^2$, $\alpha \in (0, 1)$
$t_{n,\alpha}$	A constant such that $\mathbb{P}(X > t_{n,\alpha}) = \alpha$, where $X \sim t_n$, $\alpha \in (0, 1)$
\xrightarrow{d}	Convergence in distribution
\xrightarrow{P}	Convergence in probability
i.i.d.	Independent and identically distributed
Stirling's approximation	$\Gamma(x+1) \approx \left(\frac{x}{e}\right)^x \sqrt{2\pi x}$ as $x \rightarrow \infty$, where Γ is the Gamma function
$[x]$	Greatest integer smaller than or equal to x
$\dim(V)$	Dimension of the vector space V
$\text{rank}(A)$	Rank of the matrix A
A^T	Transpose of the matrix A
f'	Derivative of the function f

Section A: Q.1 – Q.10 Carry ONE mark each.

Q.1 Let $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ be sequences given by

$$a_n = \left[\frac{n^2}{n+1} \right] \text{ and } b_n = \frac{n^2}{n+1} - a_n.$$

Then

MCQ1

- (A) $\{a_n\}_{n \geq 1}$ converges and $\{b_n\}_{n \geq 1}$ diverges
- (B) $\{a_n\}_{n \geq 1}$ diverges and $\{b_n\}_{n \geq 1}$ converges
- (C) Both $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ diverge
- (D) Both $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ converge

Q.2 Let a, b, c be real numbers with $b \neq c$. Define the matrix

$$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

Then the number of characteristic roots of M that are real is

MCQ1

(A) 3

(B) 2

(C) 1

(D) 0

Q.3 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous odd function that is not identically zero. Further, suppose that f is a periodic function. Define

$$g(x) = \int_0^x f(t) dt .$$

Then

MCQ1

- (A) g is odd and not periodic
- (B) g is odd and periodic
- (C) g is even and not periodic
- (D) g is even and periodic

Q.4 Suppose Z_1, Z_2, \dots, Z_{128} are i.i.d. $\text{Bin}(1, 0.5)$ random variables. Define

$$\mathbf{X} = (Z_1, Z_2, \dots, Z_{64})^T \text{ and } \mathbf{Y} = (Z_{65}, Z_{66}, \dots, Z_{128})^T.$$

Then the value of $\text{Var}(\mathbf{X}^T \mathbf{Y})$ is

MCQ1

- (A) 4
- (B) 8
- (C) 12
- (D) 16

- Q.5 Let X_1, X_2, X_3 be i.i.d. $\text{Bin}(1, \theta)$ random variables. Consider the problem of testing the null hypothesis $H_0 : \theta = \frac{1}{2}$ against the alternative hypothesis $H_1 : \theta = \frac{1}{4}$ based on X_1, X_2, X_3 . Then the power of the most powerful test of size 0.125 is

MCQ1

- (A) 0
- (B) $\frac{1}{64}$
- (C) $\frac{27}{64}$
- (D) $\frac{7}{8}$

Q.6 Suppose X is a $\text{Poisson}(\lambda)$ random variable. Define $Y = (-1)^X X$. Then the expected value of Y is

MCQ1

(A) $-\lambda e^{-2\lambda}$

(B) $-\lambda e^{-\lambda}$

(C) $\lambda e^{-2\lambda}$

(D) $-\lambda$

Q.7 Let $\{Y_k\}_{k \geq 1}$ be a sequence of i.i.d. $\text{Bin}(1, p)$ random variables, where $0 < p < 1$ is an unknown parameter. Let \hat{p}_n be the maximum likelihood estimator of p based on Y_1, Y_2, \dots, Y_n . It is claimed that:

$$(I) \quad \frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} N(0,1) \text{ as } n \rightarrow \infty$$

$$(II) \quad \frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} \xrightarrow{d} N(0,1) \text{ as } n \rightarrow \infty$$

Which of the following statements is correct?

MCQ1

- (A) (I) is correct and (II) is incorrect
- (B) (I) is incorrect and (II) is correct
- (C) Both (I) and (II) are correct
- (D) Both (I) and (II) are incorrect

Q.8 Let X be a continuous random variable with probability density function $f(x)$.

Consider the problem of testing the null hypothesis

$$H_0 : f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

against the alternative hypothesis

$$H_1 : f(x) = \begin{cases} 2x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the power of the most powerful size α test, where $0 < \alpha < 1$, based on a single sample, is

MCQ1

(A) $\alpha(1-\alpha)$

(B) $\alpha(2-\alpha)$

(C) $1-\alpha$

(D) α

Q.9 Suppose $X \sim N(0,4)$ and $Y \sim N(0,9)$ are independent random variables. Then the value of $\mathbb{P}(9X^2 + 4Y^2 < 6)$ is

MCQ1

(A) $1 - e^{-1/4}$

(B) $1 - e^{-1/12}$

(C) $1 - e^{-1/6}$

(D) $1 - e^{-1/9}$

Q.10 Let X be a single sample from a continuous distribution with probability density function

$$f(x; \theta) = \begin{cases} \frac{2(\theta - x)}{\theta^2} & \text{if } 0 < x < \theta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. For $0 < \alpha < 0.05$, a $100(1 - \alpha)\%$ confidence interval for θ based on X is

MCQ1

(A) $\left(\frac{X}{1 - \sqrt{\frac{\alpha}{2}}}, \frac{X}{1 - \sqrt{1 - \frac{\alpha}{2}}} \right)$

(B) $\left(\frac{X}{1 - \sqrt{\alpha}}, \frac{X}{1 - \sqrt{1 - \alpha}} \right)$

(C) $\left(\left(1 - \sqrt{1 - \frac{\alpha}{2}}\right) X, \left(1 - \sqrt{\frac{\alpha}{2}}\right) X \right)$

(D) $\left(\frac{\alpha}{2} X, \left(1 - \frac{\alpha}{2}\right) X \right)$

Section A: Q.11 – Q.30 Carry TWO marks each.

Q.11 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x + \pi \cos x$. Then the number of solutions of the equation $f(x) = 0$ is

MCQ2

(A) 1

(B) 2

(C) 3

(D) 4

Q.12 A fair die is thrown three times independently. The probability that 4 is the maximum value that appears among these throws is equal to

MCQ2

(A) $\frac{8}{27}$

(B) $\frac{1}{216}$

(C) $\frac{37}{216}$

(D) $\frac{1}{2}$

Q.13 Let A be an $n \times n$ matrix. Which of the following statements is NOT necessarily true?

MCQ2

- (A) If $\text{rank}(A^5) = \text{rank}(A^6)$, then $\text{rank}(A^6) = \text{rank}(A^7)$
- (B) If $\text{rank}(A) = n$, then it is possible to obtain a singular matrix by suitably changing a single entry of A
- (C) If $\text{rank}(A) = n$, then $\text{rank}(A + A^T) \geq \frac{n}{2}$
- (D) If $\text{rank}(A) < n$, then it is possible to obtain a nonsingular matrix by suitably changing $n - \text{rank}(A)$ entries of A

Q.14 Let V be a subspace of \mathbb{R}^{10} . Suppose A is a 10×10 matrix with real entries. Let $A^k(V) = \{A^k \mathbf{x} : \mathbf{x} \in V\}$ for $k \geq 1$ and $A(V) = A^1(V)$. Which one of the following statements is NOT true?

MCQ2

- (A) If A is nonsingular, then $\dim(V) = \dim(A(V))$ necessarily holds
- (B) It is possible that A is singular and $\dim(V) = \dim(A(V))$
- (C) If $\text{rank}(A) = 8$, then $\dim(A(V)) \geq \dim(V) - 2$ necessarily holds
- (D) If $\dim(V) = \dim(A(V)) = \dim(A^2(V)) = \dots = \dim(A^5(V))$, then $\dim(A^6(V)) = \dim(V)$ necessarily holds

Q.15 A function $f : (0,1) \rightarrow \mathbb{R}$ is said to have property \mathcal{I} if, for any $0 < x_1 < x_2 < 1$ and for any c between $f(x_1)$ and $f(x_2)$, there exists $y \in [x_1, x_2]$ such that $f(y) = c$. Consider the following statements:

- (I) If $g : (0,1) \rightarrow \mathbb{R}$ satisfies property \mathcal{I} , then g is necessarily continuous.
- (II) If $h : (0,1) \rightarrow \mathbb{R}$ is differentiable, then h' necessarily satisfies property \mathcal{I} .

Then

MCQ2

- (A) (I) is correct and (II) is incorrect
- (B) (I) is incorrect and (II) is correct
- (C) Both (I) and (II) are correct
- (D) Both (I) and (II) are incorrect

Q.16 Suppose f is a polynomial of degree n with real coefficients, and A is an $n \times n$ matrix with real entries satisfying $f(A) = 0$. Consider the following statements:

- (I) If $f(0) \neq 0$, then A is necessarily nonsingular.
- (II) If $f(0) = 0$, then A is necessarily singular.

Then

MCQ2

- (A) (I) is correct and (II) is incorrect
- (B) (I) is incorrect and (II) is correct
- (C) Both (I) and (II) are correct
- (D) Both (I) and (II) are incorrect

Q.17 Let X_1 and X_2 be i.i.d. $N(0, \sigma^2)$ random variables. Define $Z_1 = X_1 + X_2$ and $Z_2 = X_1 - X_2$. Then which one of the following statements is NOT correct?

MCQ2

(A) Z_1 and Z_2 are independently distributed

(B) Z_1 and Z_2 are identically distributed

(C) $\mathbb{P}\left(\frac{Z_1}{Z_2} < 1\right) = 0.5$

(D) $\frac{Z_1}{Z_2}$ and $Z_1^2 + Z_2^2$ are independently distributed

- Q.18 Consider a circle C with unit radius and center at $A = (0, 0)$. Let $B = (1, 0)$. Suppose $\Theta \sim U(0, \pi)$ and $D = (\cos \Theta, \sin \Theta)$. Note that the angle $\angle DAB = \Theta$. Then the expected area of the triangle ABD is

MCQ2

(A) $\frac{1}{\pi}$

(B) $\frac{2}{\pi}$

(C) $\frac{1}{2\pi}$

(D) 1

Q.19 Suppose $Y \sim U(0,1)$ and the conditional distribution of X given $Y = y$ is $\text{Bin}(6, y)$, for $0 < y < 1$. Then the probability that $(X + 1)$ is an even number is

MCQ2

(A) $\frac{3}{7}$

(B) $\frac{1}{2}$

(C) $\frac{4}{7}$

(D) $\frac{5}{14}$

Q.20 Let X be an $\text{Exp}(\lambda)$ random variable. Suppose $Y = \min\{X, 2\}$. Let F_X and F_Y denote the distribution functions of X and Y respectively. Then which of the following statements is true?

MCQ2

- (A) F_Y is a continuous function
- (B) $F_Y(y)$ is discontinuous at $y = 2$
- (C) $F_Y(t) \leq F_X(t)$ for all $t \in \mathbb{R}$
- (D) $\mathbb{E}(Y) > \mathbb{E}(X)$

Q.21 Let X_1, X_2, \dots, X_5 be i.i.d. $N(0, \sigma^2)$ random variables. Suppose c is such that

$$\mathbb{E} \left(c \sqrt{\sum_{i=1}^5 X_i^2} \right) = \sigma.$$

Then the value of c is

MCQ2

(A) $\sqrt{\frac{9\pi}{128}}$

(B) $\sqrt{\frac{9\pi}{64}}$

(C) $\sqrt{\frac{9}{128\pi}}$

(D) $\sqrt{\frac{3}{64\pi}}$

Q.22 Let X_1, X_2, \dots, X_n be a random sample from a continuous distribution with probability density function

$$f(x; \theta) = \frac{1}{2\theta} e^{-\frac{|x|}{\theta}}, \quad x \in \mathbb{R},$$

where $\theta > 0$ is an unknown parameter. The critical region for the uniformly most powerful test for testing the null hypothesis $H_0 : \theta = 2$ against the alternative hypothesis $H_1 : \theta > 2$ at level α , where $0 < \alpha < 1$, is

MCQ2

(A) $\left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : 2 \sum_{i=1}^n |x_i| < \chi_{2n, 1-\alpha}^2 \right\}$

(B) $\left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : 2 \sum_{i=1}^n |x_i| > \chi_{2n, \alpha}^2 \right\}$

(C) $\left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i| < \chi_{2n, 1-\alpha}^2 \right\}$

(D) $\left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i| > \chi_{2n, \alpha}^2 \right\}$

Q.23 Let (X, Y) have the $N_2(0, 0, 1, 1, 0.25)$ distribution. Then the correlation coefficient between e^X and e^{2Y} is

MCQ2

(A)
$$\frac{e^3 - e^{5/2}}{(e^5(e-1)(e^4-1))^{1/2}}$$

(B)
$$\frac{e^3 - e^{5/2}}{(e^4(e-1)(e^5-1))^{1/2}}$$

(C)
$$\frac{e^2 - e^{5/2}}{(e^5(e-1)(e^4-1))^{1/2}}$$

(D)
$$\frac{e^{3/2} - e^{5/2}}{(e^5(e^2-1)(e^4-1))^{1/2}}$$

Q.24 Let $\{X_k\}_{k \geq 1}$ be a sequence of i.i.d. $U(-1,1)$ random variables. Suppose

$$Y_n = \sqrt{3n} \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^4}.$$

Then $\{Y_n\}_{n \geq 1}$ converges in distribution as $n \rightarrow \infty$ to a

MCQ2

- (A) $N(0,1)$ random variable
- (B) random variable degenerate at 0
- (C) $N(0,25)$ random variable
- (D) $N(0,0.04)$ random variable

Q.25 If $(X, Y) \sim N_2(0, 0, 1, 1, 0.5)$, then the value of $\mathbb{E}(e^{-XY})$ is

MCQ2

(A) $\frac{2}{\sqrt{5}}$

(B) $\frac{2}{\sqrt{3}}$

(C) $\frac{1}{\sqrt{2}}$

(D) $\frac{1}{2}$

Q.26 Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be a random sample from a $N_2(0, 0, 1, 1, \rho)$ distribution, where ρ is an unknown parameter. Which of the following statements is NOT correct?

MCQ2

- (A) $\left(\sum_{i=1}^n X_i^2, \sum_{i=1}^n Y_i^2, \sum_{i=1}^n X_i Y_i \right)$ is a sufficient statistic for ρ
- (B) $\left(\sum_{i=1}^n X_i^2, \sum_{i=1}^n Y_i^2, \sum_{i=1}^n X_i Y_i \right)$ is not a minimal sufficient statistic for ρ
- (C) $\sum_{i=1}^n X_i^2$ is an ancillary statistic
- (D) $\left(\sum_{i=1}^n X_i^2, \sum_{i=1}^n Y_i^2 \right)$ is an ancillary statistic

Q.27 Let $(Y_1, Y_2, Y_3) \in \{0, 1, \dots, n\}^3$ be a discrete random vector having joint probability mass function

$$\mathbb{P}(Y_1 = y_1, Y_2 = y_2, Y_3 = y_3) = \begin{cases} \frac{n!}{y_1! y_2! y_3!} p^{y_1} (2p)^{y_2} (1-3p)^{y_3} & \text{if } y_1 + y_2 + y_3 = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq p \leq \frac{1}{3}$ is an unknown parameter. Assume the convention $0^0 = 1$. The maximum likelihood estimator of p is denoted by \hat{p} . Which of the following statements is correct?

MCQ2

- (A) $\mathbb{E}(\hat{p}) > p$
- (B) \hat{p} is an unbiased estimator of p , but not the uniformly minimum variance unbiased estimator of p
- (C) \hat{p} is the uniformly minimum variance unbiased estimator of p
- (D) $\mathbb{E}(\hat{p}) < p$

Q.28 Let X_1, X_2, \dots, X_n be i.i.d. $N(0,1)$ random variables, where $n > 3$. If

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \text{ and } S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2,$$

then $\text{Var}\left(\frac{\bar{X}}{S}\right)$ is equal to

MCQ2

(A) $\frac{(n-3)}{n(n-1)}$

(B) $\frac{(n-1)}{n(n-3)}$

(C) $\frac{(n-1)}{n(n-2)}$

(D) $\frac{(n-2)}{n(n-1)}$

Q.29 Suppose (X_i, Y_i) , $i = 1, 2, \dots, 200$, are i.i.d. random vectors each having joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{25\pi} & \text{if } x^2 + y^2 \leq 25, \\ 0 & \text{otherwise.} \end{cases}$$

Let M be the cardinality of the set $\{i \in \{1, 2, \dots, 200\} : X_i^2 + Y_i^2 \leq 0.25\}$. Then

$\mathbb{P}(M \geq 1)$ is closest to

MCQ2

(A) $\Phi(0.5)$

(B) $1 - e^{-1}$

(C) $\Phi(3)$

(D) $1 - e^{-2}$

Q.30 Let $\{X_n\}_{n \geq 1}$ be a sequence of random variables, where $X_n \sim \text{Bin}(n, p_n)$ with $p_n \in (0, 1)$. Which of the following conditions implies that $X_n \xrightarrow{d} 0$ as $n \rightarrow \infty$?

MCQ2

- (A) $\lim_{n \rightarrow \infty} p_n = 0$
- (B) $\lim_{n \rightarrow \infty} \mathbb{P}(X_n = k) = 0$ for each $k \in \mathbb{N}$
- (C) $\lim_{n \rightarrow \infty} \mathbb{E}(X_n) = 0$
- (D) $\sup_{n \geq 1} \text{Var}(X_n) < \infty$

Section B: Q.31 – Q.40 Carry TWO marks each.

Q.31 Let $\{x_n\}_{n \geq 1}$ be a sequence given by

$$x_n = \frac{2}{3} \left(x_{n-1} + \frac{2}{x_{n-1}} \right), \text{ for } n \geq 2,$$

with $x_1 = -10$. Then which of the following statement(s) is/are correct?

MSQ2

- (A) $\{x_n\}_{n \geq 1}$ converges
- (B) $\{x_n\}_{n \geq 1}$ diverges
- (C) $x_{2025} - x_{2024}$ is positive
- (D) $x_{2025} - x_{2024}$ is negative

Q.32 Let $a, b \in \mathbb{R}$. Consider the system of linear equations

$$\begin{aligned}x + y + 3z &= 5, \\ a x - y + 4z &= 11, \\ 2x + b y + z &= 3.\end{aligned}$$

Then which of the following statements is/are correct?

MSQ2

- (A) There are finitely many pairs (a, b) such that the system has a unique solution
- (B) There are finitely many pairs (a, b) such that the system has no solution
- (C) There are finitely many pairs (a, b) such that the system has infinitely many solutions
- (D) If $a = b = 1$, the system has no solution

Q.33 Suppose $f : (0, \infty) \rightarrow (0, \infty)$ is continuously differentiable. Assume further that $\lim_{x \rightarrow \infty} f(x) = 0$. Which of the following statements is/are necessarily true?

MSQ2

- (A) $\lim_{x \rightarrow \infty} f'(x)$ exists and is equal to 0
- (B) $\limsup_{x \rightarrow \infty} f'(x) = 0$
- (C) $\liminf_{x \rightarrow \infty} f'(x) = 0$
- (D) $\liminf_{x \rightarrow \infty} |f'(x)| = 0$

Q.34 The joint moment generating function of (X, Y) is given by

$$M_{X,Y}(s,t) = \left(\frac{1}{4} + \frac{1}{2}e^s + \frac{1}{4}e^t \right)^2, \quad (s,t) \in \mathbb{R}^2.$$

Then which of the following statements is/are correct?

MSQ2

(A) $\mathbb{E}(X) = 1$

(B) $\mathbb{E}(Y^2) = \frac{3}{8}$

(C) $\text{Cov}(X, Y) = -\frac{1}{4}$

(D) $\text{Var}(X) = \frac{1}{2}$

Q.35 The joint probability density function of X and Y is given by

$$f(x, y) = \begin{cases} c(x+y) & \text{if } 0 \leq x, y \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

for some constant c . Which of the following statements is/are correct?

MSQ2

- (A) $c=1$
- (B) X and Y are independent
- (C) The probability density function of X is $g(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$
- (D) $X+Y$ has a probability density function

Q.36 Let X_1, X_2, \dots, X_{20} be a random sample from a $N(\mu, \sigma^2)$ population. Suppose

$$P = \frac{1}{10} \sum_{i=1}^{10} X_i \text{ and } Q = \frac{1}{9} \sum_{i=1}^{10} (X_i - P)^2.$$

Then which of the following statements is/are correct?

MSQ2

(A) $\frac{X_{11} + P - X_{12} - X_{20}}{\sqrt{Q}} \sim \sqrt{\frac{31}{10}} t_9$

(B) $\frac{P - X_{15}}{\sqrt{9Q + (X_{18} - \mu)^2}} \sim \frac{\sqrt{11}}{10} t_{10}$

(C) $\frac{(X_{12} - X_{20})^2}{Q} \sim 2 F_{1,9}$

(D) $\frac{P - X_{14}}{\sqrt{Q}} \sim \frac{\sqrt{11}}{10} t_9$

Q.37 Let X_1, X_2, \dots, X_n , where $n > 1$, be a random sample from a $N(\theta, \theta)$ distribution, where $\theta > 0$ is an unknown parameter. Suppose

$$T_n = \frac{1}{n} \sum_{i=1}^n X_i \text{ and } S_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - T_n)^2.$$

Then which of the following statements is/are correct?

MSQ2

- (A) $T_n S_n^2$ is a consistent estimator for θ^2
- (B) $T_n^4 S_n^{-4}$ is a consistent estimator for θ^2
- (C) $\left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2 \right)$ is a complete statistic
- (D) $\sum_{i=1}^n X_i^2$ is a complete sufficient statistic for θ

Q.38 Let X_1, X_2, \dots, X_n , where $n > 1$, be a random sample from a continuous distribution with probability density function

$$f(x; \theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Then which of the following statistics is/are sufficient for θ ?

MSQ2

- (A) (X_1, X_2, \dots, X_n)
- (B) $(X_{(1)}, X_{(2)}, \dots, X_{(n)})$, where $X_{(r)}$ is the r^{th} order statistic, $r = 1, \dots, n$
- (C) $\sum_{i=1}^n X_i$
- (D) $\prod_{i=1}^n X_i$

Q.39 A simple linear regression model $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, with $x_i = (-1)^i$ for $i = 1, 2, \dots, 20$, is fitted. The random error variables ϵ_i are uncorrelated with mean 0 and finite variance $\sigma^2 > 0$. Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the least squares estimators of β_0 and β_1 respectively. Let \hat{Y}_i be the fitted value of the i^{th} response variable Y_i , for $i = 1, \dots, 20$. Which of the following statements is/are correct?

MSQ2

- (A) $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = 0$
- (B) $\text{Var}(\hat{\beta}_0) = \text{Var}(\hat{\beta}_1)$
- (C) $\text{Var}(\hat{\beta}_0) = \text{Cov}(\hat{Y}_i, \hat{\beta}_0)$ for all $i = 1, \dots, 20$
- (D) $\text{Var}(\hat{\beta}_1) = \text{Cov}(\hat{Y}_i, \hat{\beta}_1)$ for all $i = 1, \dots, 20$

Q.40 Let Y_1, Y_2, \dots, Y_n be i.i.d. discrete random variables from a population with probability mass function

$$\mathbb{P}(Y = y; \theta) = \begin{cases} \theta(1-\theta)^y & \text{if } y \in \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \theta \leq 1$ is an unknown parameter. Assume the convention $0^0 = 1$.

If $\hat{\theta}$ is the method of moments estimator of θ , then which of the following statements is/are correct?

MSQ2

- (A) $\hat{\theta}$ is also the maximum likelihood estimator of θ
- (B) $\hat{\theta}$ is an unbiased estimator of θ
- (C) $\hat{\theta}$ is a consistent estimator of θ
- (D) $\frac{1}{\hat{\theta}}$ is an unbiased estimator of $\frac{1}{\theta}$

Section C: Q.41 – Q.50 Carry ONE mark each.

Q.41 Let A be a 7×7 real matrix with $\text{rank}(A) = 1$. Suppose the trace of A^2 is 2025.

Let the characteristic polynomial of A be written as

$$\sum_{n=0}^7 a_n x^n.$$

Then $\sum_{n=0}^7 |a_n|$ is _____

(answer in integer)

NAT1

Q.42 The radius of convergence of the power series

$$\sum_{n \geq 1} \frac{n!}{n^n} x^n$$

is equal to _____

(round off to 2 decimal places)

NAT1

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Q.43 Let

$$f(x) = x \sin\left(\frac{\pi}{2x}\right), \quad x > 0.$$

Then

$$\lim_{h \rightarrow 0} \frac{1}{\pi^2 h^2} [3f(1) - 2f(1+h) - f(1-2h)]$$

is equal to _____

(round off to 2 decimal places)

NAT1

- Q.44 Let $\{X_k\}_{k \geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) = \mu$ and $\text{Var}(X_1) = \sigma^2 < \infty$. Suppose c is a constant that does not depend on n such that

$$\frac{c}{n} \sum_{i=1}^n (X_{2i} - X_{2i-1})^2$$

is a consistent estimator of σ^2 . Then c is equal to _____
(round off to 2 decimal places)

NAT1

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Q.45 Let X_1, X_2, \dots, X_{10} be i.i.d. $U(0, \theta)$ random variables, where $\theta > 0$ is unknown. For testing the null hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta = 0.9$, consider a test that rejects H_0 if

$$X_{(10)} = \max\{X_1, X_2, \dots, X_{10}\} < 0.8.$$

Then the probability of type I error of the test is equal to _____

(round off to 2 decimal places)

NAT1

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Q.46 Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_{100}, Y_{100})$ be i.i.d. discrete random vectors each having joint probability mass function

$$\mathbb{P}(X = x, Y = y) = \frac{e^{-(1+x)\lambda} ((1+x)\lambda)^y}{y!} p^x (1-p)^{1-x}, \quad x \in \{0, 1\}, \quad y \in \mathbb{N} \cup \{0\},$$

where $\lambda > 0$ and $0 < p < 1$ are unknown parameters. If the observed values of

$\sum_{i=1}^{100} X_i$ and $\sum_{i=1}^{100} Y_i$ are 54 and 521 respectively, the maximum likelihood

estimate of λ is equal to _____

(round off to 2 decimal places)

NAT1

Q.47 Let X and Y be i.i.d. random variables with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $Z = \min\{X, Y\}$, then $\mathbb{E}(Z)$ is equal to _____

(answer in integer)

NAT1

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Q.48 The joint probability density function of the random vector (X, Y, Z) is given by

$$f(x, y, z) = \begin{cases} \frac{1}{x y} & \text{if } 0 < z < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the value of $\mathbb{P}(X > 5Y)$ is equal to _____

(round off to 2 decimal places)

NAT1

JAM 2025

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Q.49 Suppose U_1 and U_2 are i.i.d. $U(0,1)$ random variables. Further, let X be a $\text{Bin}(2, 0.5)$ random variable that is independent of (U_1, U_2) . Then

$$36 \mathbb{P}(U_1 + U_2 > X)$$

is equal to _____

(answer in integer)

NAT1

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- Q.50 A drawer contains 5 pairs of shoes of different sizes. Assume that all 10 shoes are distinguishable. A person selects 5 shoes from the drawer at random. Then the probability that there are exactly 2 complete pairs of shoes among these 5 shoes is equal to _____
(round off to 2 decimal places)

NAT1

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Section C: Q.51 – Q.60 Carry TWO marks each.

- Q.51 Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 12\}$ be a circle in the plane. Let (a, b) be the point on C which minimizes the distance to the point $(1, 2)$. Then $b - a$ is _____ (round off to 2 decimal places)

NAT2

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- Q.52 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^3 + 2x^2 - 15x$ and $g(x) = x$ respectively. Let x_0 be the smallest strictly positive number such that $f(x_0) = 0$. Then the area of the region enclosed by the graphs of f and g between the lines $x = 0$ and $x = x_0$ is _____
(round off to 2 decimal places)

NAT2

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Q.53 Let V be the volume of the region

$$\left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + \frac{z^2}{4} \leq 1 \text{ and } |z| \leq 1 \right\}.$$

Then $\frac{V}{\pi}$ is equal to _____

(round off to 2 decimal places)

NAT2

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- Q.54 Suppose $X_1, X_2, \dots, X_{10}, Y_1, Y_2, \dots, Y_{10}$ are independent random variables, where $X_i \sim N(0, \sigma^2)$ and $Y_i \sim N(0, 3\sigma^2)$ for $i = 1, 2, \dots, 10$. The observables are D_1, \dots, D_{10} , where D_i denotes the Euclidean distance between the points $(X_i, Y_i, 0)$ and $(0, 0, 5)$ for $i = 1, 2, \dots, 10$. If the observed value of $\sum_{i=1}^{10} D_i^2$ is equal to 1050, then the method of moments estimate of σ^2 is equal to _____
(answer in integer)

NAT2

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- Q.55 Consider a sequence of independent Bernoulli trials with success probability $p = \frac{1}{7}$. Then the expected number of trials required to get two consecutive successes for the first time is equal to _____
(answer in integer)

NAT2

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- Q.56 Let X be a real valued random variable with $\mathbb{E}(X) = 1$, $\mathbb{E}(X^2) = 4$,
 $\mathbb{E}(X^4) = 16$. Then $\mathbb{E}(X^3)$ is equal to _____
(answer in integer)

NAT2

JAM 2025

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JAM 2025

Q.57 Let X_1, X_2, X_3, X_4 be a random sample from a continuous distribution with probability density function

$$f(x; \theta) = \begin{cases} 2\theta^2 x^{-3} & \text{if } \theta < x < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter. It is known that

$X_{(1)} = \min\{X_1, X_2, X_3, X_4\}$ is a complete sufficient statistic for θ . If the

observed values are $x_1 = 15, x_2 = 11, x_3 = 10, x_4 = 17$, the uniformly minimum

variance unbiased estimate of θ^2 is equal to _____

(answer in integer)

NAT2

Q.58 Let X be a random variable with probability density function

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let δ denote the conditional expectation of X given that $X \leq \frac{1}{2}$.

Then the value of 80δ is equal to _____

(answer in integer)

NAT2

JAM 2025

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- Q.59 Let X_1, X_2, \dots, X_7 be i.i.d. continuous random variables with median θ . If $X_{(1)} < X_{(2)} < \dots < X_{(7)}$ are the corresponding order statistics, then $\mathbb{P}(X_{(2)} > \theta)$ is equal to _____
(round off to 3 decimal places).

NAT2

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Q.60 Suppose (X, Y) has the $N_2(3, 0, 4, 1, 0.5)$ distribution. Then
 $4 \text{Cov}(X + Y, Y^3)$ is equal to _____

(answer in integer)

NAT2

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