MATHEMATICAL SCIENCES

Time : 2 Hours *Maximum Marks: 200*

Instructions for the Candidates

1. If $u_1 = \sqrt{6}$ and $u_{n+1} = \sqrt{6 + u_n}$ for $n \geq 1$, then the sequence $\{u_n\}$ converges to

    (A) 0

    (B) 3

    (C) 6

    (D) 9

2. Let A be a matrix with complex entries. If A is Hermitian as well as unitary and α is an eigenvalue of A, then

    (A) α can be any real number.

    (B) $\alpha = 1$ or $\alpha = -1$.

    (C) α can be any complex number of absolute value 1.

    (D) α will have unique value 1.

3. If x is a feasible solution to the primal *max* $z = cx$, subject to $Ax \leq b$, $x \geq 0$ and v be the feasible solution to the dual problem *min* $u = b^T v$ subject to $A^T v \geq c^T$, $v \geq 0$, then

    (A) $cx = b^T v$

    (B) $cx \geq b^T v$

    (C) $cx \leq b^T v$

    (D) $c^T x = bv$

4. If in the Rolle's theorem, the third condition (*i.e.* $f(a) = f(b)$) is replaced by $\dfrac{f(a)}{a} = \dfrac{f(b)}{b}$ $(a, b \neq 0)$, then

    (A) the conclusion of the Rolle's theorem remains true.

    (B) $f'(\xi) = \xi$, $a < \xi < b$.

    (C) $\dfrac{f'(\xi)}{f(\xi)} = \xi$, $a < \xi < b$.

    (D) $\xi f'(\xi) = f(\xi)$.

5. For $\alpha > 0$ and $\beta > 0$, the function

$$f(x) = \begin{cases} x^\alpha \sin \dfrac{1}{x^\beta}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is of bounded variation in [0, 1] if

    (A) $\alpha > \beta$

    (B) $\alpha < \beta$

    (C) $1 + \alpha < \beta$

    (D) $1 - \alpha > \beta$

6. Inverse point of $z = 1 + i$ with respect to $|z| = 2$ is

    (A) $(2, 2)$

    (B) $(\sqrt{2}, \sqrt{2})$

    (C) $(1, 1)$

    (D) $(-1, -1)$

7. The set of all matrices of the form $\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$, where $x, y \in \mathbb{Q}$ under the operation of matrix addition and multiplication is

    (A) a ring with identity.

    (B) a commutative ring.

    (C) a non-commutative ring with zero divisors.

    (D) a ring without zero divisors but with unity.

8. Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ be linearly independent, and

$\delta_1 = x_2 y_3 - y_2 x_3$, $\delta_2 = x_1 y_3 - y_1 x_3$, $\delta_3 = x_1 y_2 - y_1 x_2$. If V is the span of x, y, then

    (A) $V = \{(u, v, w) \mid \delta_1 u - \delta_2 v + \delta_3 w = 0\}$

    (B) $V = \{(u, v, w) \mid -\delta_1 u + \delta_2 v + \delta_3 w = 0\}$

    (C) $V = \{(u, v, w) \mid \delta_1 u + \delta_2 v - \delta_3 w = 0\}$

    (D) $V = \{(u, v, w) \mid \delta_1 u + \delta_2 v + \delta_3 w = 0\}$

[Please Turn Over]

9. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij} = 3$ for all i and j. Then nullity of A is

(A) $n - 1$

(B) $n - 3$

(C) n

(D) 0

10. The eigenvalues of the boundary value problem

$$\frac{d^2 y}{dx^2} + \lambda y = 0, \ x \in [0, \pi]$$

with $y(0) = 0$ and $y(\pi) - y'(\pi) = 0$ are given by

(A) $\lambda = k_n^2$, where k_n, $(n = 1, 2, 3, ...)$ are the roots of $k + \tan(k\pi) = 0$.

(B) $\lambda = n^2$, $n = 1, 2, 3, ...$

(C) $\lambda = (n\pi)^2$, $n = 1, 2, 3, ...$

(D) $\lambda = k_n^2$, where k_n, $(n = 1, 2, 3, ...)$ are the roots of $k - \tan(k\pi) = 0$.

11. Let $f \in C^1 [-1, 1]$, then $\displaystyle\lim_{n \to \infty} \sum_{k=1}^{n} f'\left(\frac{k}{3n}\right)$ is equal to

(A) $3\left[f\left(\dfrac{1}{3}\right) - f(0) \right]$

(B) $3\left[f\left(\dfrac{1}{3}\right) + f(0) \right]$

(C) $\left[f\left(\dfrac{1}{3}\right) - f(0) \right]$

(D) $\left[f\left(\dfrac{1}{3}\right) + f(0) \right]$

12. Which of the following functions is Riemann integrable on $[0, 1]$?

(A) $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

(B) $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

(C) $f(x) = \displaystyle\lim_{n \to \infty} \cos^{2n}(24\pi x)$

(D) $f(x) = \displaystyle\lim_{n \to \infty} \cos^{3n}(21\pi x)$

13. For $0 < |z| < 1$, $\dfrac{1 + 2z}{z^2 + z^3}$ can be expressed as

(A) $\dfrac{1}{z^3} + \dfrac{1}{z^2} + \dfrac{1}{z} - 1 + z - z^2 + z^3 - ...$

(B) $\dfrac{1}{z^2} + \dfrac{1}{z} - 1 + z - z^2 + z^3 - ...$

(C) $\dfrac{1}{z} - 1 + z - z^2 + z^3 - ...$

(D) $\dfrac{1}{z} - 1 - z - z^2 - z^3 - ...$

14. The number of elements of a principal integral domain can be

(A) 15

(B) 25

(C) 35

(D) 36

15. The number of elements in the field $\dfrac{\mathbb{Z}[i]}{\langle 2 + i \rangle}$ is

(A) 2

(B) 3

(C) 5

(D) ∞

16. Let $X = \mathbb{N}$, the set of positive integers. Consider the metrics d_1 and d_2 on X given by

$$d_1(m, n) = |m - n|, \ m, n \in X$$

$$d_2(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|, \ m, n \in X.$$

Let X_1 and X_2 denote the metric spaces (X, d_1) and (X, d_2) respectively. Then

(A) X_1 is complete.

(B) X_2 is complete.

(C) X_1 is totally bounded.

(D) X_2 is not totally bounded.

17. Which of the following statements is correct?

(A) Sylow–2 subgroup of group $S_3 \times \mathbb{Z}_{10}$ is normal.

(B) Sylow–3 subgroup of group $S_4 \times \mathbb{Z}_5$ is normal.

(C) Sylow–p subgroup of group $G_1 \times G_2$ is normal, where G_1 and G_2 are finite abelion groups.

(D) Sylow–3 subgroup of group $S_4 \times \mathbb{Z}_3$ is normal.

18. Consider the Banach space $C[0, \pi]$ with the sup norm. The norm of the linear functional $l : C[0, \pi] \to \mathbb{R}$, given by $l(f) = \int_0^\pi f(x) \sin^2 x \, dx$ is

(A) 1

(B) $\dfrac{\pi}{2}$

(C) π

(D) 2π

19. Let $X = \{1, 2, 3, 4, 5\}$ and $\tau = \{\varphi, \{1\}, \{1, 2\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 2, 3, 4\}, X\}$ be a topology on X. Which is not a τ-neighbourhood of 3?

(A) $\{1, 3, 4\}$

(B) $\{1, 3, 4, 5\}$

(C) $\{1, 3\}$

(D) $\{1, 2, 3, 4\}$

20. Let $X = \mathbb{R}$ with cofinite topology. Then X is a

(A) first countable space

(B) T_1 space

(C) regular space

(D) normal space

21. The solution of $x\, u_x + y\, u_y = 0$ is of the form

(A) $f\left(\dfrac{x}{y}\right)$

(B) $f(x + y)$

(C) $f(x - y)$

(D) $f(xy)$

22. Let $N = a_m 10^m + a_{m-1} 10^{m-1} + ... + a_1 10 + a_0$ be the decimal expansion of the positive integer N, $0 \le a_n \le 10$, ($n = 1, 2, ..., m$) and let $T = a_0 - a_1 + a_2 + ... + (-1)^n a_n$. Then $11/N$ if and only if

(A) $11/T + 1$

(B) $11/T$

(C) $11/T - 1$

(D) $T/11$

23. A rigid body, symmetrical about the z-axis, has one point fixed on this axis. Let $\vec{\omega} = \omega_x \hat{i} = \omega_y \hat{j} + \omega_z \hat{k}$ be the angular velocity of the body. Then

(A) ω_x and ω_y are constant.

(B) ω_z is constant.

(C) ω_z must be zero.

(D) $\omega_x^2 + \omega_y^2 = \omega_z^2$.

24. The stress field for an isotropic solid is given by

$\tau_{11} = \tau_{22} = \tau_{33} = \tau_{12} = 0$

$\tau_{13} = -\mu\alpha x_2, \; \tau_{23} = \mu\alpha x_1,$

$\alpha \ne 0$ is a constant and μ is the Lame's constant. The strain energy function is equal to

(A) $\dfrac{1}{4\mu\alpha^2}\left(x_1^2 + x_2^2\right)$

(B) $\dfrac{\mu\alpha^2}{4}\left(x_1^2 + x_2^2\right)$

(C) $\mu\alpha^2\left(x_1^2 + x_2^2\right)$

(D) $4\mu\alpha^2\left(x_1^2 + x_2^2\right)$

[Please Turn Over]

25. A circular cylinder of radius *a* is fixed across a stream of fixed velocity *U* and circulation *k* round the cylinder. Then the maximum velocity is

(A) $2U + \dfrac{k}{\pi a}$

(B) $U + \dfrac{k}{\pi a}$

(C) $3U + \dfrac{k}{2\pi a}$

(D) $2U + \dfrac{k}{2\pi a}$

26. A poset in which every pair of elements has both *lub* and *glb* is termed as

(A) sublattice

(B) trail

(C) work

(D) lattice

27. For the curve $x = a \tan \theta$, $y = a \cot \theta$, $z = a\sqrt{2}\ \log(\tan \theta)$, the radius of curvature is

(A) $\dfrac{2\sqrt{2}\ a}{\sin^2 \theta}$

(B) $\dfrac{2\sqrt{2}\ a}{\sin^2 2\theta}$

(C) $\dfrac{2\sqrt{2}\ a}{\sin \theta}$

(D) $\dfrac{2\sqrt{2}\ a}{\sin 2\theta}$

28. Trapezoidal rule for evaluation of $\displaystyle\int_a^b f(x)\,dx$ requires the interval (*a*, *b*) to be divided into

(A) $2n$ subintervals of equal width.

(B) $2n + 1$ subintervals of equal width.

(C) any number of subintervals of equal width.

(D) $3n$ subintervals of equal width.

29. The variational problems of extremizing the functional $I[y(x)] = \displaystyle\int_1^3 y(3x - y)\,dx$, $y(3) = 4\dfrac{1}{2}$, $y(1) = 1$ has

(A) a unique solution.

(B) exactly two solutions.

(C) infinitely many solutions.

(D) no solution.

30. Let $y(s)$ be the solution of the integral equation
$$y(s) = s + \int_0^1 s\,u^2\,y(u)\,du.$$
Then $y(3)$ is given by

(A) 1

(B) 2

(C) 3

(D) 4

31. Let *f* and *g* be measurable real-valued function on $\mathbb{R}$ such that
$$\int_{-\infty}^{\infty} (f(x)^2 + g(x)^2)\,dx = 2\int_{-\infty}^{\infty} f(x)g(x)\,dx.$$
If $E = \{x \in \mathbb{R}\ !\ f(x) \neq g(x)\}$, then which of the following statements is not true?

(A) *E* is a non-empty set.

(B) *E* is measurable.

(C) *E* has Lebesgue measure zero.

(D) For almost all $x \in \mathbb{R}$, we have $f(x) = 0$ and $g(x) = 0$.

32. If the partial differential equation
$(x - 1)^2 u_{xx} - (y - 2)^2 u_{yy} + 2xu_x + 2yu_y + 2xyu = 0$ is parabolic in $S \subseteq \mathbb{R}^2$, then *S* is

(A) $\{(x, y) \in \mathbb{R}^2\ !\ x = 1 \text{ or } y = 2\}$

(B) $\{(x, y) \in \mathbb{R}^2\ !\ x = 1 \text{ and } y = 2\}$

(C) $\{(x, y) \in \mathbb{R}^2\ !\ x = 1\}$

(D) $\{(x, y) \in \mathbb{R}^2\ !\ y = 2\}$

33. If f is a differentiable function on the real line such that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} f'(x) = a$, then

(A) $a = 0$

(B) $a \ne 0$ but $|a| < 1$

(C) $a > 1$

(D) $a < -1$

34. Let the joint m.g.f. of (X, Y) be given by

$M_{X,Y}(s,t) = a(e^{s+t} + 1) + b(e^s + e^t), a > 0, b > 0$ and $a + b = \dfrac{1}{2}$. Then the correlation between X and Y is

(A) $4a$

(B) $4a - 1$

(C) $4b$

(D) $4b - 1$

35. If X is a non-negative integer-valued random variable and $P(s) = E(s^X)$, then $\int\limits_0^1 P(s^2)\, ds$ equals

(A) $E(X^2)$

(B) $E(\sqrt{X})$

(C) $E\left(\dfrac{1}{X+1}\right)$

(D) $E\left(\dfrac{1}{2X+1}\right)$

36. A coin is tossed repeatedly until a head turns up. Let X be the number of tosses it takes until this happens. The probability of getting a head is p. If it is known that tail has fallen 3 times, the probability that head will fall within next two tosses is

(A) $(1-p)^2 p^3$

(B) $(1-p)^3 p^2$

(C) $(2-p)p$

(D) $(2-p)p^2$

37. Let X be a random variable with $E(X) = 0$ and $\text{Var}(X) = \sigma^2$. Then

(A) $\Pr(X > x) \le \dfrac{\sigma^2}{\sigma^2 + x^2}, x > 0$

(B) $\Pr(X > x) \ge \dfrac{\sigma^2}{\sigma^2 + x^2}, x > 0$

(C) $\Pr(X > x) \le \dfrac{\sigma^2}{\sigma^2 + x^2}, x < 0$

(D) $\Pr(X > x) \le \dfrac{x^2}{\sigma^2 + x^2}, x < 0$

38. In estimating the location parameter of a distribution under the transformation of location, if $\bar{x}$ and $\tilde{x}$ denote, respectively, the sample mean and sample median, then

(A) both $\bar{x}$ and $\tilde{x}$ are location equivariant.

(B) $\bar{x}$ is location equivariant, but $\tilde{x}$ is not.

(C) $\tilde{x}$ is location equivariant, but $\bar{x}$ is not.

(D) neither $\bar{x}$ nor $\tilde{x}$ is location equivariant.

39. Let X have a binomial $(5, p)$ distribution and p has prior distribution Beta $(2, 3)$. An observation $X = 4$ is obtained from this set up. Then the posterior distribution of p is

(A) Beta $(6, 4)$

(B) Uniform $(0, 1)$

(C) Beta $(3, 2)$

(D) Uniform $(2, 3)$

40. For a two-sample location problem based on independent samples of size n_1 and n_2 respectively, the Wilcoxon–Mann–Whitney test statistic for testing equality of two distribution functions F_1 and F_2 has (mean, variance) under the null hypothesis as

(A) $\left(\dfrac{n_1(n_1 + n_2 + 1)}{2}, \dfrac{n_1 n_2(n_1 + n_2)}{4} \right)$

(B) $\left(\dfrac{n_2(n_1 + n_2 + 1)}{2}, \dfrac{n_1 n_2(n_1 + n_2 + 1)}{12} \right)$

(C) $\left(\dfrac{n_1 n_2(n_1 + n_2 + 1)}{2}, \dfrac{n_1 n_2(n_1 + n_2 + 1)}{12} \right)$

(D) $\left(\dfrac{n_1(n_1 + n_2 + 1)}{2}, \dfrac{n_1 n_2(n_1 + n_2 + 1)}{12} \right)$

41. The frequency distribution of the radius of a large number of spherical balls is bell-shaped and symmetric about its median. Then the frequency distribution of the volume of those balls will be

    (A) symmetric bell-shaped

    (B) negatively skewed

    (C) positively skewed

    (D) J-shaped

42. Consider a linear model with observations Y_1, Y_2, Y_3, Y_4 such that

$E(Y_1) = \theta_1 + \theta_2 + \theta_3$

$E(Y_2) = \theta_1$

$E(Y_3) = \theta_2$

$E(Y_4) = \theta_1 - \theta_2 - \theta_3$.

Which of the following statements is/are true?

1. $\theta_2 + \theta_3$ is not estimable.
2. θ_1, θ_2 and θ_3 are all estimable.
3. $\theta_1 + \theta_2 + \theta_3$ is estimable.
4. Y_2 is the BLUE of θ_1.

    (A) 1 only

    (B) 1 and 3 only

    (C) 2 and 3 only

    (D) 2 and 4 only

43. Suppose that events occur following Poisson process with average time between occurrence of events calculated at 60 minutes. Given that one event just occurred at 12 noon, what is the probability that the next occurrence of the event will take place after 2 p.m.?

    (A) e^{-120}

    (B) e^{-12}

    (C) e^{-2}

    (D) e^{-4}

44. For a life table, which one is correct?

    (A) $L_x \cong l_x - \dfrac{1}{2}d_x \, , \, e_x^{\,0} = \dfrac{T_x}{l_x}$

    (B) $L_x \cong l_x + \dfrac{1}{2}d_x \, , \, e_x^{\,0} = \dfrac{T_x}{l_x}$

    (C) $L_x \cong l_x + \dfrac{1}{2}d_x \, , \, e_x^{\,0} = \dfrac{T_x}{L_x}$

    (D) $L_x \cong l_x - \dfrac{1}{2}d_x \, , \, e_x^{\,0} = \dfrac{T_x}{L_x}$

45. Let Λ be the likelihood ratio statistic for testing $H_0 : \underset{\sim}{\mu} = \underset{\sim}{\mu}_0$ against $H : \underset{\sim}{\mu} \neq \underset{\sim}{\mu}_0$ for a p-variate normal distribution $N_p\left(\underset{\sim}{\mu}, \Sigma\right)$ based on a random sample of size n, and T^2 be the corresponding Hotellings T^2 statistic. Then

    (A) $\Lambda^{n/2}$ is a linear function of T^2.

    (B) $\Lambda^{2/n}$ is a linear function of T^2.

    (C) $\Lambda^{-2/n}$ is a linear function of T^2.

    (D) $\Lambda^{-n/2}$ is a linear function of T^2.

46. A time series whose properties do not depend on the time at which the series is observed

    (A) is stationary.

    (B) is non-stationary.

    (C) has significant trend.

    (D) has seasonality.

47. A simple random sample of size 3 units is drawn with replacement from a population of N units. The probability that the sample consists of all three distinct units is

    (A) $\dfrac{1}{N^2}$

    (B) $\dfrac{1}{N^3}$

    (C) $\dfrac{(N-1)^2}{N^3}$

    (D) $\dfrac{(N-1)(N-2)}{N^2}$

48. In a $M \mid M \mid 1$ queuing model, an analyst informs that the service rate is 8/hour and $P(N \geq 2) = \dfrac{1}{4}$ (where N is the number of customers in the system). Then the arrival rate is

    (A) 2/hour

    (B) 8/hour

    (C) 10/hour

    (D) 4/hour

49. The steady state system state distribution of the G | M | 1 queue is

(A) Poisson distribution

(B) Uniform distribution

(C) Binomial distribution

(D) Geometric distribution

50. In the context of two-person zero sum game, which of the following is correct?

(A) All mixed strategies can be obtained as a particular case of some fixed pure strategy.

(B) All pure strategies can be obtained as a particular case of mixed strategies.

(C) Pure strategies are not related to mixed strategies and vice versa.

(D) All the above are incorrect.

51. Consider a multiple regression model with r regressors and the response variable Y. Suppose $\hat{Y}$ is the fitted value of Y, R^2 is the coefficient of determination and R_a^2 is the adjusted coefficient of determination. Then which of the following statements is/are true?

1. R^2 always increases if an additional regressor is included in the model.

2. R_a^2 always increases if an additional regressor is included in the model.

3. $R^2 > R_a^2$ for all r.

4. The regression line of $\hat{Y}$ on Y has a negative slope.

(A) 2 only

(B) 1, 3 and 4 only

(C) 1 and 3 only

(D) 3 and 4 only

52. In the E O Q model, assuming that order quantity (Q) arrives all at once just when desired, planned shortages are not allowed and a constant demand rate of 'a' unit per unit time; the optimum cycle time is

(A) $\sqrt{\dfrac{2\,ah}{k}}$

(B) $\sqrt{\dfrac{2\,k}{ah}}$

(C) $\sqrt{\dfrac{ah}{2\,k}}$

(D) $\sqrt{\dfrac{k}{2\,ah}}$

where h is the unit holding cost and k is the set up cost.

53. Let $N(t)$ denote the number of customers in a M | M | 1 | 7 queuing system with arrival rate = service rate = 3/hour.

Which of the following is true?

(A) The steady state probability distribution of number of customers in the system cannot be determined since arrival rate is equal to service rate.

(B) $\lim_{t\to\infty} P\big(N(t)=3\big) = \lim_{t\to\infty} P\big(N(t)=4\big)$

(C) $\lim_{t\to\infty} P\big(N(t)=0\big) = 0$

(D) All of the above

54. The Hazard rate of the 2 parameter Weibull distribution (with shape parameter β and scale parameter λ) is

(A) $\dfrac{\beta}{\lambda}\left(\dfrac{t}{\lambda}\right)^{\beta-1}$

(B) $\left(\dfrac{t}{\lambda}\right)^{\beta-1}$

(C) $e^{-\left(\frac{t}{\lambda}\right)^{\beta}}$

(D) $1 - e^{-\left(\frac{t}{\lambda}\right)^{\beta}}$

55. In a time series forecasting problem, if the seasonal indices for quarters 1, 2 and 3 are 0·80, 0·90 and 0·95 respectively, what can you say about the seasonal index of quarter 4?

    (A) It will be less than 1.

    (B) It will be greater than 1.

    (C) It will be equal to 1.

    (D) Data are insufficient to conclude.

56. Suppose $\{X_n\}$ is a sequence of pairwise independent random variables with $P\left(X_n = -n^\alpha\right) = P\left(X_n = n^\alpha\right) = \dfrac{1}{2}$. Then WLLN holds for $\{X_n\}$ if

    (A) $0 < \alpha < \dfrac{1}{2}$

    (B) $\dfrac{1}{2} < \alpha < 1$

    (C) $\alpha > 1$

    (D) $0 < \alpha < \infty$

57. In the usual 3σ control chart for fraction defectives, under normality, the sample size n required to detect with probability 0·5 a shift in the process fraction defective from p_0 to p_1 $(> p_0)$ is

    (A) $\dfrac{3 p_0 (1 - p_0)}{(p_1 - p_0)^2}$

    (B) $\dfrac{9 p_0 (1 - p_0)}{(p_1 - p_0)}$

    (C) $\dfrac{9 p_1 (1 - p_1)}{p_1 - p_0}$

    (D) $\dfrac{9 p_0 (1 - p_0)}{(p_1 - p_0)^2}$

58. For the Markov chain with state space $\{1, 2, 3, 4\}$ and the transition probability matrix

$$\begin{pmatrix} \dfrac{1}{4} & \dfrac{1}{2} & 0 & \dfrac{1}{4} \\[2mm] \dfrac{1}{3} & 0 & \dfrac{1}{3} & \dfrac{1}{3} \\[2mm] \dfrac{1}{3} & \dfrac{1}{2} & 0 & \dfrac{1}{6} \\[2mm] \dfrac{1}{3} & \dfrac{1}{3} & \dfrac{1}{3} & 0 \end{pmatrix}$$

    consider the following statements:

  1. The chain is irreducible.

  2. All the states are recurrent.

  3. State 4 is transient.

    Which of the above statement(s) is/are true?

    (A) 1 only

    (B) 2 only

    (C) 1 and 3 only

    (D) 1 and 2 only

59. Let D be a BIBD with parameters b, v, r, k and λ. Which of the following statements is/are true?

  1. D is connected if $k \geq 2$.

  2. The covariance between the BLUE's of a pair of orthogonal treatment contrasts is zero.

  3. The efficiency factor of D relative to a RBD with replication r is strictly smaller than unity.

    (A) 1 only

    (B) 1 and 2 only

    (C) 3 only

    (D) 2 and 3 only

60. The joint pdf of X, Y is

$$f(x, y) = \frac{3x + y}{4} \cdot \exp(-x - y), x > 0, y > 0.$$

    Then the covariance between X and Y is

    (A) $\dfrac{3}{16}$

    (B) $-\dfrac{3}{16}$

    (C) $\dfrac{5}{16}$

    (D) $-\dfrac{5}{16}$

61. Thirty students wrote a mathematics examination having a maximum of 50 marks. The marks distribution is given in the following stem and leaf plot:

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0 | 9
1 | 2 2 5
2 | 0 1 3 3 3 5 8 8 9
3 | 0 0 1 3 6 6 7 9
4 | 0 2 2 4 4 4 7 8
5 | 0
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The median mark is

(A) 30.5

(B) 30.0

(C) 25.0

(D) 28.5

62. Let $X_1, X_2, ..., X_n$ be a random sample from $f(x, \theta) = \theta \, exp \, (-\theta x)$, $x > 0$. Then the UMVUE's of θ and $\dfrac{1}{\theta}$ are respectively

(A) $\bar{X}, \dfrac{1}{\bar{X}}$

(B) $\dfrac{1}{\bar{X}}, \bar{X}$

(C) $\bar{X}, \dfrac{n-1}{n\bar{X}}$

(D) $\dfrac{n-1}{n\bar{X}}, \bar{X}$

63. Let $X_1, X_2, ..., X_n$ be a random sample from Uniform $\left(\mu - 3\sqrt{\sigma}, \mu + 3\sqrt{\sigma}\right)$. Then MLEs of μ and σ are respectively [$X_{(r)}$ stands for r-th order statistic]

(A) $\bar{X}$ and $\dfrac{1}{n}\sum\limits_{i=1}^{n}\left(X_i - \bar{X}\right)^2$

(B) $X_{(1)}$ and $X_{(n)}$

(C) $\dfrac{X_{(1)} + X_{(n)}}{2}$ and $\dfrac{X_{(n)} - X_{(1)}}{2\sqrt{3}}$

(D) $\dfrac{X_{(1)} + X_{(n)}}{2\sqrt{3}}$ and $\dfrac{X_{(n)} + X_{(1)}}{2}$

64. The asymptotic lines on the surface $x = 3u(1 + v^2) - u^3, y = 3v(1 + u^2) - v^3, z = 3u^2 - 3v^2$ are

(A) $u + v = $ constant

(B) $u - v = $ constant

(C) $\pm \dfrac{u}{2v} = $ constant

(D) Both (A) and (B)

65. Consider the series $x_{n+1} = \dfrac{x_n}{2} + \dfrac{9}{8 x_n}$, $n = 0, 1, 2, ...$ with $x_0 = 0.5$ obtained from the Newton–Raphson method. The series converges to

(A) 1.5

(B) $\sqrt{2}$

(C) 1.6

(D) 1.4

66. The extremal of the functional

$$I[y(x)] = \int_a^b \frac{\sqrt{1 + y'^2}}{y} \, dx$$

are

(A) circles

(B) parabolas

(C) ellipses

(D) straight lines

67. The homogeneous integral equation

$$y(x) - \lambda \int_0^1 (3x - 2)t \, y(t) \, dt = 0 \text{ has}$$

(A) one characteristic number.

(B) two characteristic numbers.

(C) three characteristic numbers.

(D) no characteristic number.

68. Let $I = [2, 3)$. J be the set of all rational numbers in the interval $[4, 6]$. K be the cantor (ternary) set, and let $L = \{7 + x \,!\, x \in K\}$. Then what is the Lebesgue measure of the set $I \cup J \cup L$?

    (A) 0

    (B) 1

    (C) 2

    (D) 3

69. Fourier cosine transform of e^{-x^2} is

    (A) $\dfrac{1}{2} e^{-\lambda^2/4}$

    (B) $\dfrac{1}{\sqrt{2}} e^{-\lambda^2/4}$

    (C) $\dfrac{1}{2} e^{-\lambda^2/2}$

    (D) $\dfrac{1}{\sqrt{2}} e^{-\lambda^2/2}$

    where λ is the transformed variable.

70. Which of the following is the Fourier transform of
$$F(x) = \begin{cases} 1 - x^2, & \text{if } |x| \le 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$
(λ is the transformed variable)?

    (A) $\dfrac{4}{\lambda^2}(\sin\lambda - \lambda\cos\lambda)$

    (B) $\dfrac{4}{\lambda^3}(\sin\lambda - \lambda\cos\lambda)$

    (C) $\dfrac{4}{\lambda^2}(\cos\lambda - \lambda\cos\lambda)$

    (D) $\dfrac{4}{\lambda^3}(\cos\lambda - \lambda\sin\lambda)$

71. Suppose $(X_1, X_2, ..., X_n)$ is a random sample from a Cauchy distribution with pdf
$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Consider the following statements:

1. X_1 and $-X_1$ have the same distribution.

2. X_1 and $\bar{X}$ have the same distribution.

3. $\bar{X}$ and $\dfrac{1}{\bar{X}}$ have the same distribution.

    Which of the above statements is/are true?

    (A) 1 only

    (B) 2 only

    (C) 1 and 2 only

    (D) All of 1, 2 and 3

72. Let X_1, X_2 and X_3 be i.i.d. random variables with common pdf $f_X(x) = exp(-x), x > 0$.

    Then the pdf of $Y = \dfrac{X_1 + X_2}{X_1 + X_2 + X_3}$ is

    (A) $f_Y(y) = 1, 0 < y < 1$

    (B) $f_Y(y) = \dfrac{3}{2}, 0 < y < \dfrac{2}{3}$

    (C) $f_Y(y) = 2y, 0 < y < 1$

    (D) $f_Y(y) = 3y^2, 0 < y < 1$

73. In terms of the incomplete beta function defined as $I_u(x,y) = \dfrac{\displaystyle\int_0^u t^{x-1}(1-t)^{y-1}\,dt}{\displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt}$, the distribution function of the r-th order statistic when the random sample of size n is drawn from the distribution function $F(x)$ is

    (A) $I_{F(x)}(r-1,\, n-r)$

    (B) $I_{F(x)}(r-1,\, n-r+1)$

    (C) $I_{F(x)}(r,\, n-r+1)$

    (D) $I_{F(x)}(r,\, n-r)$

74. In a series of houses actually invaded by small pox, 70% of the inhabitants are attacked and 85% have been vaccinated. The percentage of the vaccinated that have been attacked cannot be smaller than

(A) 64.7%

(B) 70.2%

(C) 80%

(D) 58.3%

75. Suppose independent samples are available from k univariate normal populations $N(\mu_1, \sigma_1^2)$, $N(\mu_2, \sigma_2^2),..., N_k(\mu_k, \sigma_k^2)$, where all parameters are unknown. Let λ_1 be LR statistic for testing equality of the k means under the assumption of homoscedasticity and λ_2 be the LR statistic for testing homoscedasticity. If λ be the LR statistic for testing equality of all the k populations, then

(A) λ cannot be computed from λ_1 and λ_2

(B) $\lambda = \sqrt{\lambda_1 \lambda_2}$

(C) $\lambda = \min(\lambda_1, \lambda_2)$

(D) $\lambda = \lambda_1 \lambda_2$

76. Directed graph is a graph in which

(A) edges do not have a direction.

(B) set of vertices are connected by directed edges.

(C) the connection between two nodes is necessarily reciprocated.

(D) the edges indicate a two-way relationship.

77. If $\vec{v} = 2t\hat{i} + x\hat{j}$ is the velocity of two dimensional fluid flow and $\psi(x, y, t)$ is the stream function, then

(A) $\psi(x, y, t) = -2yt + \dfrac{1}{2}x^2 + g(t)$,

(B) $\psi(x, y, t) = -2t + \dfrac{1}{2}x^2 + g(t)$,

(C) $\psi(x, y, t) = -2yt + \dfrac{1}{2}x + g(t)$,

(D) $\psi(x, y, t) = -2y + \dfrac{1}{2}x^2 + g(t)$,

where $g(t)$ is an arbitrary function of t.

78. The deformation of a body is defined by the displacement components $u_1 = k(3x_1^2 + x_2)$, $u_2 = k(2x_2^2 + x_3)$, $u_3 = k(4x_3 + x_1)$, where k is a positive constant. The extension of a line element that passes through the point $(1, 1, 1)$ in the direction $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ is

(A) $\dfrac{16k}{3}$

(B) $\dfrac{17k}{3\sqrt{3}}$

(C) $\dfrac{17k}{3}$

(D) $\dfrac{22k}{3\sqrt{3}}$

79. Let H, T and V denote the Hamiltonian, the kinetic energy and the potential energy respectively of a mechanical system at time t. If H depends on t explicitly, the $\dfrac{\partial H}{\partial t}$ is equal to

(A) $\dfrac{\partial T}{\partial t} + \dfrac{\partial V}{\partial t}$

(B) $\dfrac{\partial T}{\partial t} - \dfrac{\partial V}{\partial t}$

(C) $\dfrac{\partial V}{\partial t} - \dfrac{\partial T}{\partial t}$

(D) $-\dfrac{\partial V}{\partial t} - \dfrac{\partial T}{\partial t}$

80. Consider the congruence $x^n \equiv 2 \pmod{13}$. This congruence has a solution for x if

(A) $n = 8$

(B) $n = 6$

(C) $n = 7$

(D) $n = 9$

81. Let us consider the following two Sturm-Liouville type ordinary differential equations:

(1) $x^2 u'' + 2xu' = x^2$, $0 \le x \le 1$ with the boundary conditions $u(0)$ is finite and $u(1) + u'(1) = 0$.

(2) $u'' + u = f(x)$, $0 \le x \le \pi$, with $u(0) = \alpha$ and $u(\pi) = \beta$.

Consider the following two statements.

(i) Green's function for the first problem exists.

(ii) Green's function for the second problem exists for any arbitrary function $f(x)$.

Then,

    (A) both the statements (i) and (ii) are correct.

    (B) only (i) is correct.

    (C) only (ii) is correct.

    (D) both (i) and (ii) are not correct.

82. Nagata–Smirnov metrization theorem of G_δ set states : A subset A of topological space X is G_δ set in X

    (A) if it is equal to the union of a countable collection of open subsets of X.

    (B) if it is equal to the intersection of a countable collection of open subsets of X.

    (C) if it is equal to the union of a uncountable collection of open subsets of X.

    (D) if it is equal to the uncountable collection of closed subsets of X.

83. Let X be the space of bounded real sequences with sup norm. Define a linear operator $T : X \to X$ by

$$T(x) = \left(\frac{x_1}{1}, \frac{x_2}{2}, \dots \right) \text{ for } x = (x_1, x_2, \dots) \in X.$$

Then

    (A) T is bounded but not one-to-one.

    (B) T is one-to-one but not bounded.

    (C) T is bounded and its inverse (from range of T) exists but not bounded.

    (D) T is bounded and its inverse (from range of T) exists and bounded.

84. Let $A : H \to H$ be any bounded linear operator on a complex Hilbert space H such that $\| Ax \| = \| A^*x \|$ for all x in H, were A^* is the adjoint of A. If there is a non-zero x in H such that $A^*(x) = (2 + 3i)x$, then A is

    (A) a unitary operator on H.

    (B) a self-adjoint operator on H but not unitary.

    (C) a self-adjoint operator on H but not normal.

    (D) a normal operator.

85. The degree of the extension of $\mathbb{Q}\left(\sqrt{2} + \sqrt[3]{2}\right)$ over the field $\mathbb{Q}\left(\sqrt{2}\right)$ is

    (A) 1

    (B) 2

    (C) 3

    (D) 6

86. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 2n, & \text{if } \dfrac{1}{2n} \le x \le \dfrac{1}{n} \\ 0, & \text{otherwise} \end{cases}.$$

Then the values of $\displaystyle\int_0^1 \lim_{n \to \infty} f_n \, d\mu$ and $\displaystyle\lim_{n \to \infty} \int_0^1 f_n \, d\mu$, where μ is the Lebesgue measure on $\mathbb{R}$, are respectively

    (A) 0 and 0

    (B) 0 and 1

    (C) 1 and 0

    (D) 1 and 1

87. Let $\langle p(x) \rangle$ denote the ideal generated by the polynomial $p(x)$ in $\mathbb{Q}[x]$. If $f(x) = x^3 + x^2 + x + 1$ and $g(x) = x^3 - x^2 + x - 1$, then

    (A) $\langle f(x) \rangle + \langle g(x) \rangle = \langle x^3 + x \rangle$

    (B) $\langle f(x) \rangle + \langle g(x) \rangle = \langle f(x)\, g(x) \rangle$

    (C) $\langle f(x) \rangle + \langle g(x) \rangle = \langle x^2 + 1 \rangle$

    (D) $\langle f(x) \rangle + \langle g(x) \rangle = \langle x^4 - 1 \rangle$

88. If g is analytic at $z = a$ and f has a pole of order k at $z = a$, then $res_{z=a} \dfrac{g(z)f'(z)}{f(z)}$ is

   (A) $kg(a)$

   (B) $-\dfrac{1}{k}g(a)$

   (C) $-kg(a)$

   (D) $-k!g(a)$

89. The radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \cos\dfrac{n\pi}{4}$ is

   (A) larger than 1.

   (B) smaller than 1.

   (C) equal to 1.

   (D) equal to 0.

90. Let $S = \{(0, 1, \alpha), (\alpha, 1, 0), (1, \alpha, 1)\}$. Then S is a basis for $\mathbb{R}^3(\mathbb{R})$ if and only if

   (A) $\alpha \neq 0$

   (B) $\alpha \neq 1$

   (C) $\alpha \neq 0$ and $\alpha^2 \neq 2$

   (D) $-1 \leq \alpha \leq 1$

91. Consider the linear programming problem:

   $max\ z = \alpha\, x_1 + x_2$
   subject to $2x_1 + x_2 \leq 6$
   $\qquad\qquad -x_1 + x_2 \leq 1$
   $\qquad\qquad x_1 + x_2 \leq 4$
   $\qquad\qquad x_1, x_2 \geq 0,$

   where α is a constant. If $(3, 0)$ is the only optimal solution, then

   (A) $\alpha < -2$

   (B) $-2 < \alpha < 0$

   (C) $\alpha > 2$

   (D) $\alpha < 2$

92. The function $f(x) = \sin\left(\dfrac{x}{2}\right)$, $x \in [\,0, \pi\,]$ is

   (A) monotonically increasing in $[\,0, \pi\,]$.

   (B) monotonically decreasing in $[\,0, \pi\,]$.

   (C) monotonically increasing in $\left[\,0, \dfrac{\pi}{2}\,\right]$ and then decreasing in $\left[\,\dfrac{\pi}{2}, \pi\,\right]$.

   (D) neither monotonically increasing nor decreasing.

93. Let $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$,

   then

   (A) Cauchy–Riemann equations are satisfied at the origin and $f'(0) = 0$.

   (B) Cauchy–Riemann equations are satisfied at the origin but $f'(0)$ does not exist.

   (C) Cauchy–Riemann equations are not satisfied at the origin.

   (D) Cauchy–Riemann equations are not satisfied at the origin and $f'(0) = 0$.

94. Which of the following functions is not harmonic?

   (A) $u(x, y) = \sin hx \cos y$

   (B) $u(x, y) = \dfrac{1}{2} \log(x^2 + y^2)$

   (C) $u(x, y) = x^2 - y^2$

   (D) $u(x, y) = x^2 + y^2$

95. In the group $(\mathbb{Z}, +)$, the subgroup generated by 2 and 7 is

   (A) $\mathbb{Z}$

   (B) $5\mathbb{Z}$

   (C) $9\mathbb{Z}$

   (D) $14\mathbb{Z}$

96. Let T be a linear transformation on $\mathbb{R}^3$ given by $T(x, y, z) = (2x, 4x - y, 3x + 3y - z)$, then $T^{-1}(x, y, z)$ is

(A) $(\frac{x}{2}, 2x - y, 7x - 3y - z)$

(B) $(x, 2x + y, 4x + 3y - z)$

(C) $(\frac{x}{2}, 4x + 3y, 7x + 3y - z)$

(D) $(x, 3x + y, 7x - 3y + z)$

97. The initial value problem $\frac{dy}{dx} = \cos(xy)$, $x \in \mathbb{R}$, $y(0) = y_0$, where y_0 is a real constant, has

(A) a unique solution.

(B) exactly two solutions.

(C) infinitely many solutions.

(D) no solution.

98. The Wronskian of the functions $f_1(x) = x^2$ and $f_2(x) = x\,|x|$ is zero

(A) for all x.

(B) for $x > 0$.

(C) for $x < 0$.

(D) for $x = 0$.

99. Which of the following improper integrals is convergent?

(A) $\int\limits_{0}^{\infty} \dfrac{x \tan^{-1} x}{(1 + x^4)^{1/3}}\, dx$

(B) $\int\limits_{e^2}^{\infty} \dfrac{dx}{x \log x}$

(C) $\int\limits_{2}^{3} \dfrac{3x^2 + 1}{x^2 - 4}\, dx$

(D) $\int\limits_{0}^{1} \dfrac{\sin x}{x^{3/2}}\, dx$

100. The bilinear transformation which maps the points $z = 1$, $z = 0$, $z = -1$, of z-plane into $\omega = i$, $\omega = 0$, $\omega = -i$ of ω-plane respectively is

(A) $\omega = iz$

(B) $\omega = z$

(C) $\omega = i(z + 1)$

(D) $\omega = -iz$

Space for Rough Work

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