HS/XII/A. Sc. Com/M/23

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MATHEMATICS

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

General Instructions :

 (i) All questions are compulsory.

 (ii) This question paper contains 36 questions divided into four Sections—A, B, C and D. Section—A comprises of 20 questions of *1* mark each, Section—B comprises of 6 questions of *2* marks each, Section—C comprises of 6 questions of *4* marks each and Section—D comprises of 4 questions of *6* marks each.

(iii) There is no overall choice. However, internal choice has been provided in 9 questions of Section—A, 5 questions of Section—B, 5 questions of Section—C and 3 questions of Section—D. You have to attempt only one of the alternatives in all such questions.

(iv) Use of calculator is not permitted.

<div align="center">SECTION—A</div>

1. Define an equivalence relation.   1

<div align="center">*Or*</div>

A relation R in the set N of natural numbers is defined as $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$. Find the range of R.   1

[P.T.O.

(2)

2. Find the principal value of $\cos^{-1}\left(\dfrac{1}{2}\right)$.   1

Or

Evaluate :   1

$$\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$$

3. Find the value of x, if

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$$ 1

Or

If $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$, find AB and BA.   1

4. What is an objective function of a linear programming problem?   1

5. Evaluate :   1

$$\int_0^{\pi/2} \cos 2x\, dx$$

Or

Evaluate :   1

$$\int_0^1 \dfrac{dx}{1+x^2}$$

(3)

6. Differentiate :   1

$$2\sqrt{\cot x}$$

Or

Find $\dfrac{dy}{dx}$, if $y = \sec(\tan x)$.   1

7. Write the order and degree of the differential equation

$$\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$$ 1

8. Prove that $y = Ax$ is a solution of the differential equation

$$xy' = y, \ (x \neq 0)$$

and A is a constant.   1

9. Show that the function $f : R \to R$ given by $f(x) = x^3$ is injective, where R is the set of real numbers.   1

Or

Show that the modulus function $f : R \to R$ given by $f(x) = |x|$ is not one-one, where R is the set of real numbers.   1

10. Find $\dfrac{dy}{dx}$, if $y = \log(\cos e^x)$.   1

Or

If $2x - 3y = \sin x$, find $\dfrac{dy}{dx}$.   1

(**4**)

11. If $P(A) = \dfrac{6}{11}$, $P(B) = \dfrac{5}{11}$ and $P(A \cup B) = \dfrac{7}{11}$, find $P(A \cap B)$.   1

12. Let E and F be events with $P(E) = \dfrac{3}{5}$, $P(F) = \dfrac{3}{10}$ and

$P(E \cap F) = \dfrac{1}{5}$. Are E and F independent?   1

13. Evaluate :   1

$$\int x^2 \left(1 - \dfrac{1}{x^2}\right) dx$$

Or

Evaluate :   1

$$\int \tan^2 x \, dx$$

14. Compute the magnitude of the following vector :   1

$$\vec{a} = \dfrac{1}{\sqrt{3}}\hat{i} + \dfrac{1}{\sqrt{3}}\hat{j} + \dfrac{1}{\sqrt{3}}\hat{k}$$

15. Evaluate :   1

$$\int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$$

16. Prove that the function

$$f(x) = x^3 + 3x^2 + 3x + 100$$

is increasing in R, where R is the set of real numbers.   1

Choose the correct answer :

17. If $\sin^{-1} x = y$, then

    (a) $0 \le y \le \pi$

    (b) $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

    (c) $0 < y < \pi$

    (d) $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$ 1

18. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then $A + A' = I$, if the value of α is

    (a) $\dfrac{\pi}{6}$

    (b) $\dfrac{\pi}{3}$

    (c) π

    (d) $\dfrac{3\pi}{2}$ 1

19. The anti-derivative of $\left(\sqrt{x}\;\dfrac{1}{\sqrt{x}}\right)$ is equal to

(a) $\dfrac{1}{3}x^{\frac{1}{3}}\quad 2x^{\frac{1}{2}}\quad c$

(b) $\dfrac{2}{3}x^{\frac{2}{3}}\quad \dfrac{1}{2}x^{2}\quad c$

(c) $\dfrac{2}{3}x^{\frac{3}{2}}\quad 2x^{\frac{1}{2}}\quad c$

(d) $\dfrac{3}{2}x^{\frac{3}{2}}\quad \dfrac{1}{2}x^{\frac{1}{2}}\quad c$ 1

Or

The rate of change of the area of a circle with respect to its radius r at r 6 cm is

(a) 10

(b) 12

(c) 8

(d) 11   1

20. The value of $\hat{i}\;(\hat{j}\;\hat{k})\quad \hat{j}\;(\hat{i}\;\hat{k})\quad \hat{k}\;(\hat{i}\;\hat{j})$ is

(a) 0

(b) 1

(c) 1

(d) 3   1

SECTION—B

21. Find x and y, if

$$2 \begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix} \qquad 2$$

Or

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, show that $|2A| = 4|A|$. $\qquad$ 2

22. Find the value of k, if the function defined by

$$f(x) = \begin{cases} kx^2 & , \text{ if } x \le 2 \\ 3 & , \text{ if } x > 2 \end{cases}$$

is continuous at $x = 2$. $\qquad$ 2

Or

Show that $f(x) = 5x - 3$ is continuous at $x = 5$. $\qquad$ 2

23. Evaluate : $\qquad$ 2

$$\int x \, e^x \, dx$$

Or

Evaluate : $\qquad$ 2

$$\int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$$

(8)

24. If $y = 500e^{7x} + 600e^{-7x}$, show that

$$\frac{d^2y}{dx^2} = 49y$$

2

25. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$.

2

Or

Show that the points $A(2, 3, 4)$, $B(-1, -2, 1)$ and $C(5, 8, 7)$ are collinear.

2

26. Find $\frac{dy}{dx}$, if

$$y = \cos^{-1}\frac{1-x^2}{1+x^2}, \quad 0 < x < 1$$

2

Or

Evaluate :

2

$$\int \frac{x^2}{1+x^6}\,dx$$

SECTION—C

27. Show that the function $f : R \to R$ defined by $f(x) = 3 - 4x$ is one-one and onto, where R is the set of real numbers.   4

Or

Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation.   4

28. Find the intervals in which the function f given by

$$f(x) = 4x^3 - 6x^2 - 72x + 30$$

is *(a)* strictly increasing and *(b)* strictly decreasing.   4

Or

Sand is pouring from a pipe at the rate of $12 \text{ cm}^3 / \text{s}$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?   4

29. Using the properties of definite integrals, evaluate

$$\int_0^{\pi/4} \log (1 + \tan x)\, dx$$ 4

Or

Using partial fraction, evaluate

$$\int \frac{1}{x^4 - 1}\, dx$$ 4

30. Solve the following homogeneous differential equation :   4

$$xdy - ydx = \sqrt{x^2 + y^2}\, dx$$

Or

Solve the following linear differential equation :

$$(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1 + x^2}$$

given $y = 0$, when $x = 1$.   4

31. If a unit vector $\vec{a}$ makes angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence the components of $\vec{a}$.   4

Or

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.   4

32. Solve the following linear programming problem graphically :   4

Maximize $Z = 4x + y$

subject to the constraints

$$x + y \le 50$$
$$3x + y \le 90$$
$$x \ge 0, y \ge 0$$

SECTION—D

33. Verify $A \,(\text{adj } A) = (\text{adj } A)\,A = |A|\,I$ for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

6

Or

Solve the following system of linear equations using matrix method :

6

$$x + y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12$$

34. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\dfrac{8}{27}$ of the volume of the sphere.

6

Or

Using definite integral, find the area of the region bounded by the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

6

35. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + (\hat{i} - 3\hat{j} + 2\hat{k})$$

and

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + (2\hat{i} + 3\hat{j} + \hat{k})$$

6

36. A manufacturer has three machine operators *A*, *B* and *C*. The first operator *A* produces 1% defective items whereas the other two operators *B* and *C* produce 5% and 7% defective items respectively. *A* is on the job for 50% of the time, *B* is on the job for 30% of the time and *C* is on the job for 20% of the time. A defective item is produced. What is the probability that it was produced by *A*?　　6

Or

Let *X* denote the number of hours you study during a randomly selected school day. The probability that *X* can take the values *x* has the following form, where *k* is a constant :

$$P(X = x) = \begin{cases} 0 \cdot 1 & , \text{ if } x = 0 \\ kx & , \text{ if } x = 1 \text{ or } 2 \\ k(5 - x) & , \text{ if } x = 3 \text{ or } 4 \\ 0 & , \text{ otherwise} \end{cases}$$

Find the value of *k*. What is the probability that you study at least 2 hours, exactly 2 hours and at most 2 hours?　　6

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